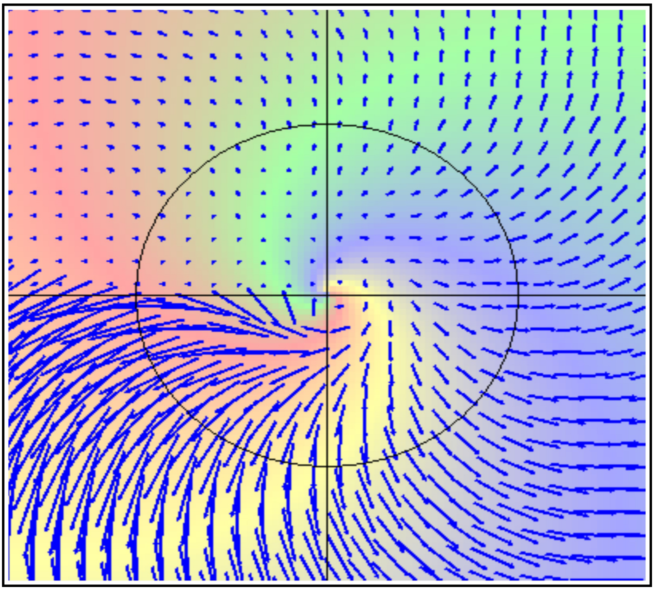


Complex Functions

Oliver Linton

CONTENTS

The Complex Plane

A complex number has the form $\boldsymbol{z} = x + \boldsymbol{i}y$ where $\boldsymbol{i}$ is the square root of minus one. The number x is called the *real* part and y is the *imaginary* part. Such a number can be conveniently represented by a point in a plane using x and y as Cartesian coordinates.

It is also often convenient to represent the same number in polar coordinates $\boldsymbol{z} = r\angle\theta$ where r is called the *modulus* of $\boldsymbol{z}$ and θ is the *argument* of $\boldsymbol{z}$.

Simple trigonometry shows that

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x)$$

(i.e. the 'angle whose tangent is' y/x.)

A complex function takes as its argument a complex number $\boldsymbol{z}$ and returns another complex number $\boldsymbol{w}$. i.e.

$$\boldsymbol{w} = f(\boldsymbol{z}) = u + \boldsymbol{i}v$$

It is the purpose of this little book to illustrate a wide variety of complex functions and to describe their characteristics in the same way the a simple graph can illustrate the properties of a real function like $y = x^2 - 1$.

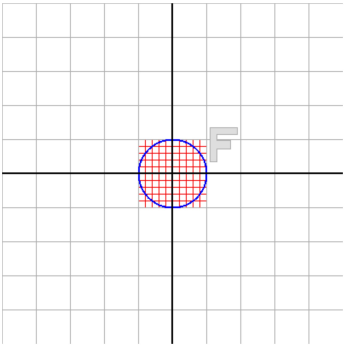
But we have a problem. To illustrate a complex function we need four dimensional graph paper because both u and v depend on x and y.

Several solutions to present themselves. Firstly we can show how a typical figure in the $\boldsymbol{z}$-plane is mapped onto the $\boldsymbol{w}$-plane.

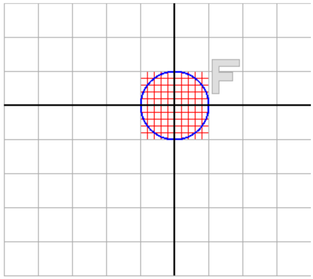
Alternatively we can use colours or shading to show how a single aspect of the function is mapped across the plane.

Thirdly we can use a 3D plot to show one aspect and colour to show the other.

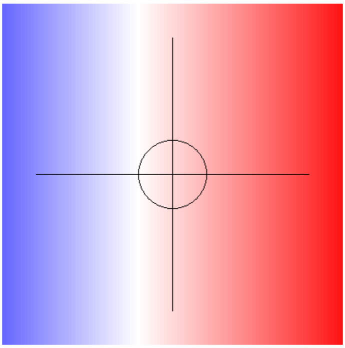
All these methods are illustrated on the following pages using the simple function $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{c}$ where $\boldsymbol{c}$ is a complex number equal to $1 + 2\boldsymbol{i}$.



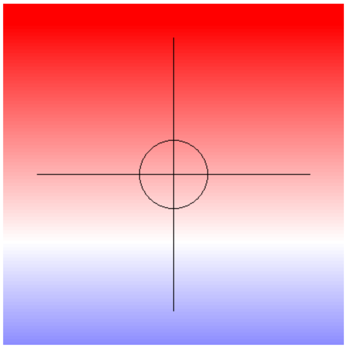
*The **z** plane*



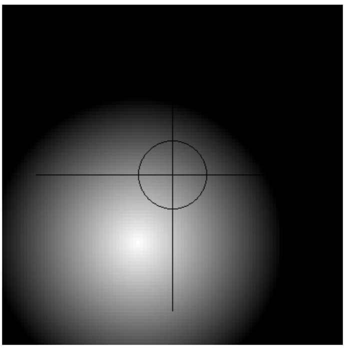
*The **z** plane is shifted one unit to the right and two units upwards.*



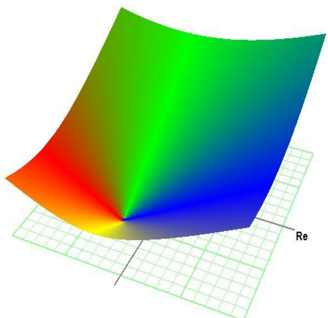
The real part of $\mathbf{w} = \mathbf{z} + 1 + 2\mathbf{i}$
Red is positive, blue negative



The imaginary part of $\mathbf{w} = \mathbf{z} + 1 + 2\mathbf{i}$
Red is positive, blue negative



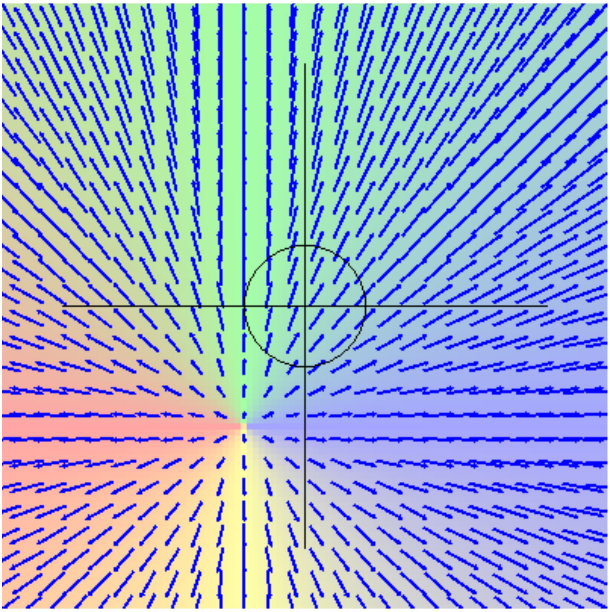
The modulus of $\mathbf{w} = \mathbf{z} + 1 + 2\mathbf{i}$



The modulus of $\mathbf{w} = \mathbf{z} + 1 + 2\mathbf{i}$ *with the argument shown in colour*

There is yet another way we can illustrate a complex function.

Instead of separating the real and imaginary parts of $\boldsymbol{w}$ we can regard $\boldsymbol{w}$ as a *vector* with length r pointing in the direction θ. At each point in a matrix of points in the $\boldsymbol{z}$ plane we can plot a small arrow whose length is proportional to r pointing in the direction θ. The result is a powerful sense of the 'flow' of the function, radiating out from the root.



The vector plot of $\boldsymbol{w} = \boldsymbol{z} + 1 + 2\boldsymbol{i}$
All the vectors radiate from the point (-1, -2)

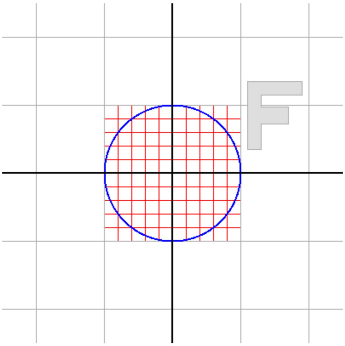
The direction of the arrows is emphasised by the use of a background colour – blue for East, green for North, red for West and yellow for South.

Another very simple function is $\boldsymbol{w} = \boldsymbol{cz}$ where $\boldsymbol{c}$ is, once again a complex number.

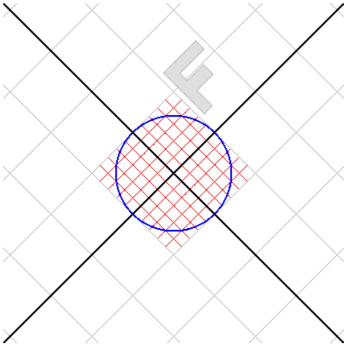
Whereas the operation of addition of two complex numbers is pretty straightforward (you just add the real and the imaginary parts of each), multiplication is a bit more interesting. To see how it works we need to use the polar forms of the two numbers not their Cartesian forms. What we do is *multiply* the *moduli* and *add* the *arguments*. i.e.

$$cz = r_c r_z \angle(\theta_c + \theta_z)$$

Lets have a look at the function $\boldsymbol{w} = \boldsymbol{cz}$ where c = $0.8\angle 45°$ (or $0.56 + 0.56\boldsymbol{i}$ in Cartesian coordinates.)

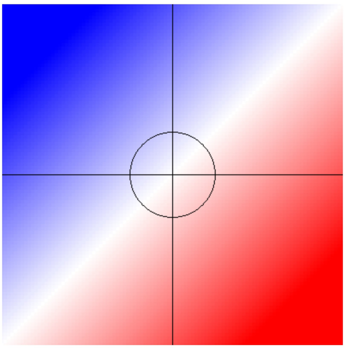


The $\boldsymbol{z}$ *plane*

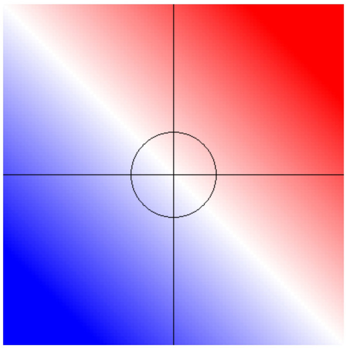


The $\boldsymbol{w}$ *plane* $\boldsymbol{w} = (0.8\angle 45°)\boldsymbol{z}$

The result is to shrink the plane to 80% and rotate it by 45° in an anticlockwise direction. Here are the real and imaginary plots. Note that they are rotated clockwise, not anti-clockwise.

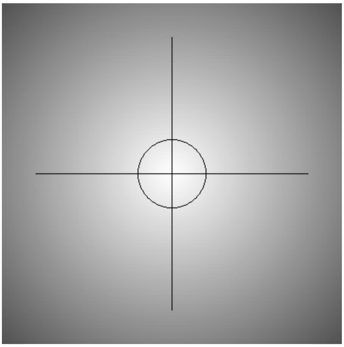


The real part of $\boldsymbol{w} = (0.8\angle 45°)\boldsymbol{z}$

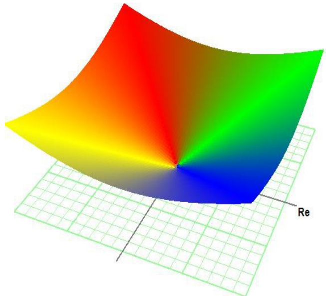


The imaginary part of $\boldsymbol{w} = (0.8\angle 45°)\boldsymbol{z}$

The modulus plot looks much the same but the argument has been rotated by 45° clockwise. (The colours are coded as follows: 0° is blue, 90° is green, 180° is red and 270° is yellow.)

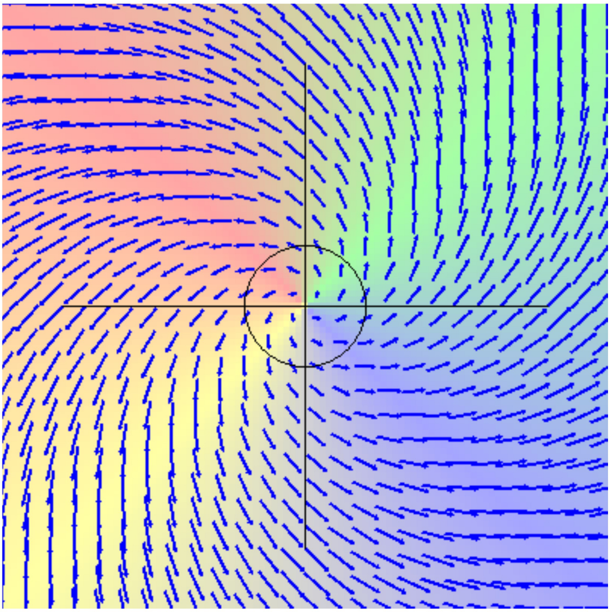


The modulus ***of w*** = (0.8∠45°)***z***



The 3D (modulus/argument) plot ***of w***

The vector plot is a bit surprising. It is not rotated by 45°; instead, *every vector* is rotated by 45°. This gives the overall flow a pleasing swirling motion around and away from the origin.

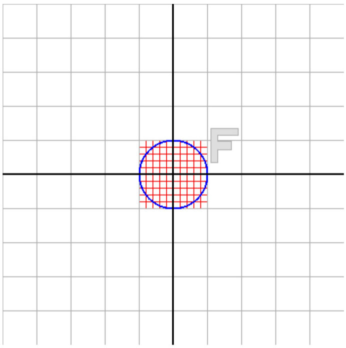


The vector plot of **w** = (0.8∠45°)**z**

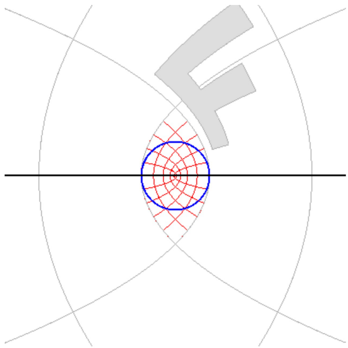
POLYNOMIAL FUNCTIONS

Things start to get really interesting with even the simplest of polynomial functions: $f(z) = z^2$.

In the last chapter we said that when two complex numbers are multiplied together, the moduli are multiplied and the arguments are added. If the two numbers are the same, the modulus of the number must be *squared* and the argument *doubled*.

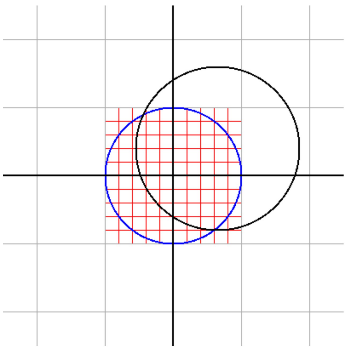


The z plane

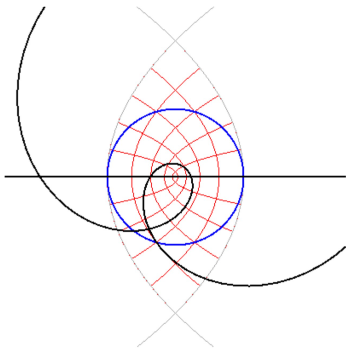


The $\mathbf{w}$ *plane* $\mathbf{w} = z^2$

Notice how letter F has been distorted by being both rotated and stretched. The horizontal and vertical lines of the original Z plane have been bent into hyperbolae.



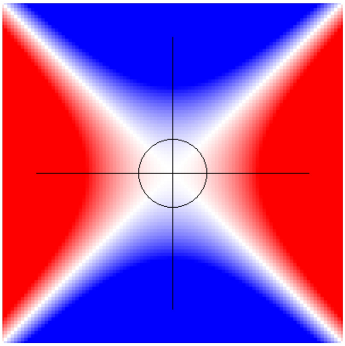
The z plane



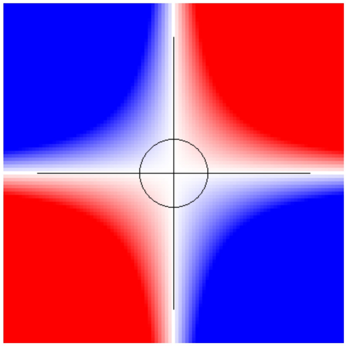
The $\mathbf{w}$ *plane* $\mathbf{w} = z^2$

Note how a circle is transformed into a cardioid which wraps twice round the origin (because of the doubling of the argument).

The real and imaginary parts of z^2 look like this:



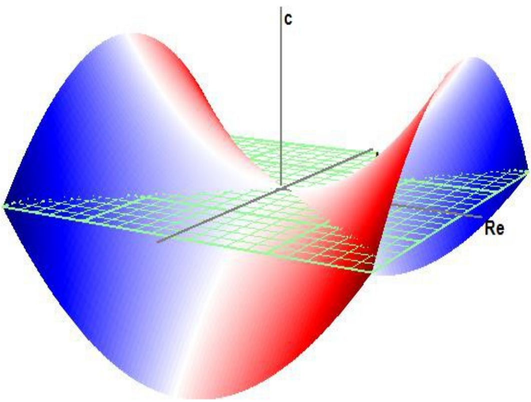
The real part of $\boldsymbol{w} = z^2$



The imaginary part of $\boldsymbol{w} = z^2$

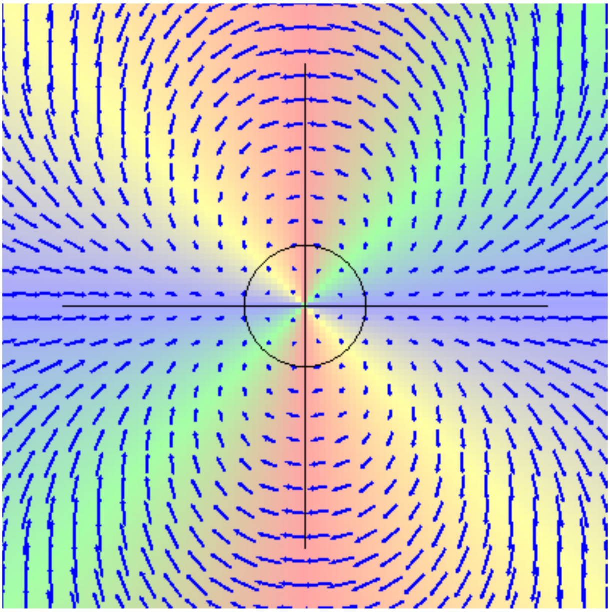
We can explain these graphs as follows. Since $\boldsymbol{z} = x + \boldsymbol{i}y$, $z^2 = x^2 - y^2 + 2\boldsymbol{i}xy$. The real part of this is $x^2 - y^2$. This will be zero whenever $x = y$ which accounts for the diagonal cross. A cross-section along the X axis (where $y = 0$) will be a positive parabola while a cross section across the Y axis will be an inverted parabola.

These two graphs can be combined into one with the real part portrayed in 3D with the imaginary part in colour. It has the familiar shape of a Pringle.



The real and imaginary parts of $\boldsymbol{w} = z^2$

But, as always, the most instructive plot is the vector plot.



The vector plot of $w = z^2$

The first and most obvious feature is the fact that the further you go away from the origin, the longer the arrows become. This reflects the fact that the modulus of z^2 is the square of the modulus of z. (Here the modulus has also been indicated in shades of grey.)

But what about the doubling of the argument? Where is that feature?

Imagine that you have a small compass which magically points in the direction of the arrows when placed in the w plane. Place the compass at the point (2,0) It will point East. Now imagine moving it anti-clockwise slowly round the origin in a circle of radius 2. When you get to the point (0,2) it will point West and when you get to (–2,0) it will have made one complete revolution. When you have completed the circle the compass will have rotated twice! We have here an example of what is called a *winding number* and in this case it has the value 2.[1]

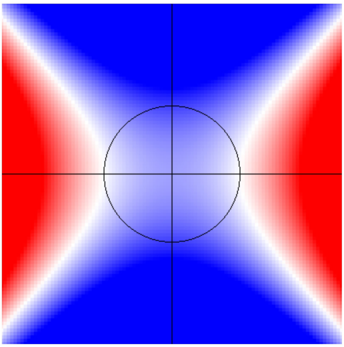
1 Strictly speaking this is the known as the *index* of the point. As you can see from the map on page 7, the image of a circle which encloses the origin winds round the

This doubling of the rate of rotation is obvious when you look at the coloured map of the argument. A complete rotation around the origin takes you from blue through green, red, yellow, blue, green, red, yellow and back to blue.

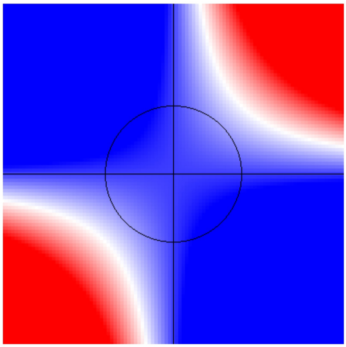
It is worth taking a little time to think how all these plots will be modified if we consider the general case where $\boldsymbol{w} = \boldsymbol{z}^2 + \boldsymbol{cz} + \boldsymbol{d}$ (where $\boldsymbol{c}$ and $\boldsymbol{d}$ are complex numbers of course).

Consider the function $\boldsymbol{w} = \boldsymbol{z}^2 + \boldsymbol{d}$ where $\boldsymbol{d} = -(1 + 2\boldsymbol{i})$. If this were an ordinary (real) function (e.g. $y = x^2 - 1$), the addition or subtraction of a constant would simply shift the parabola up or down the Y axis. In exactly the same way, the complex function retains its (four dimensional) shape but is shifted in such a way as to pass though the (four dimensional) point $(0, \boldsymbol{d})$. If you find it difficult to thinking in four dimensions, here are some plots of the function $\boldsymbol{w} = \boldsymbol{z}^2 - (1 + 2\boldsymbol{i})$.

First the real and imaginary parts.



The real part of $\boldsymbol{w} = \boldsymbol{z}^2 - (1 + 2\boldsymbol{i})$



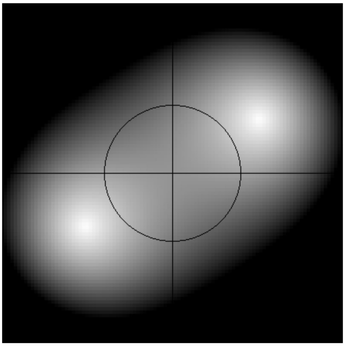
The imaginary part of $\boldsymbol{w} = \boldsymbol{z}^2 - (1 + 2\boldsymbol{i})$

The crosses have been replaced by rectangular hyperbolae because the 'Pringle' (of the real part) has been shifted down so that its saddle is at -1. Likewise the 'imaginary Pringle' is shifted down so that its saddle is at $-2\boldsymbol{i}$.

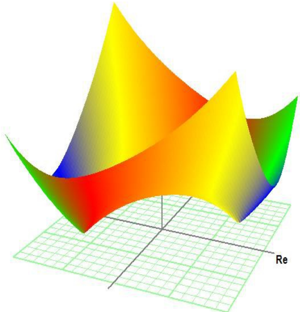
The modulus plot shows the original single root at the origin has split

origin twice. In general we can say that the winding number of the image is equal to the index of the point. It is this fact which (partially) justifies my talking about the winding number of a point.

into two roots at $\pm\sqrt{(1 + 2\boldsymbol{i})}$.

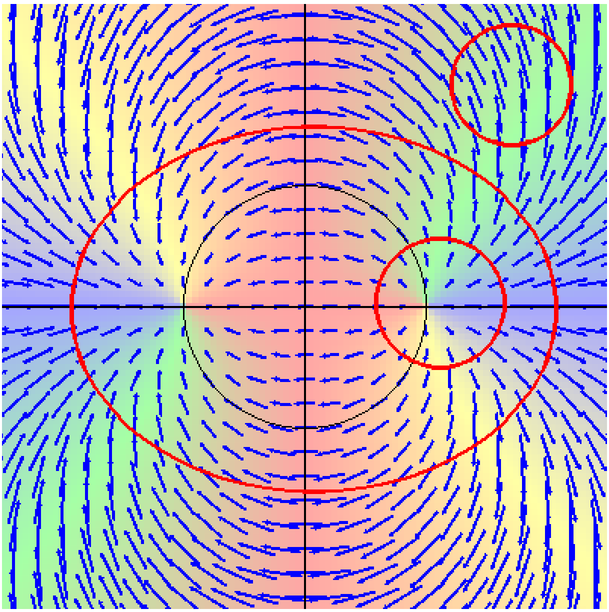


The modulus of $\boldsymbol{w} = z^2 - (1 + 2\boldsymbol{i})$



The 3D plot of of $\boldsymbol{w} = z^2 - (1 + 2\boldsymbol{i})$

Before we look at the vector plot of this function, take a look at the plot of the simpler function $\boldsymbol{w} = \boldsymbol{z}^2 - 1$

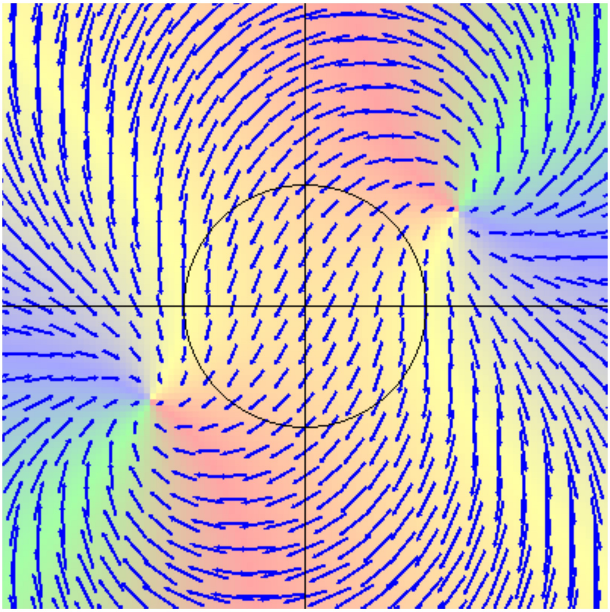


The vector plot of $\boldsymbol{w} = z^2 - 1$

Now take your imaginary plotting compass and trace its rotation as you move it round the small circle that encloses one of the roots in an anti-clockwise direction. It rotates once, anti-clockwise, doesn't it. Now trace it round the large ellipse which encloses two roots. You will not be surprised to find that it rotates twice. This seems to suggest the following rule: in any closed circuit, a plotting compass will rotate n times *in the same direction* where n is the number of roots which the circuit encloses. If this is so, then the compass should not rotate at all if the circuit doesn't contain any roots. Check this with the small circle in the top right.

It follows that every point in the plane has an associated winding number (i.e. the winding number of an infinitesimal loop enclosing the point). For the great majority of points the winding number is zero but the two roots are obviously different. They are examples of what I call *exceptional points* which I define as any point whose winding number is not zero. We shall meet other kinds of exceptional point soon.

Now lets look at the function $\boldsymbol{w} = \boldsymbol{z}^2 - (1 + 2\boldsymbol{i})$ again:



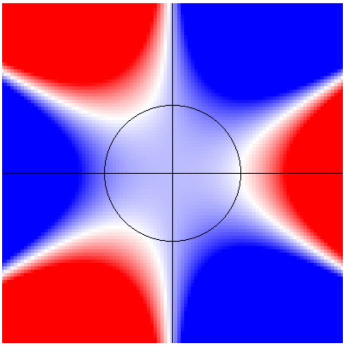
The vector plot of $\boldsymbol{w} = \boldsymbol{z}^2 - (1 + 2\boldsymbol{i})$

The roots have moved but the winding numbers are the same as before.

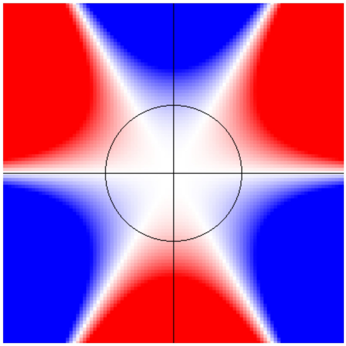
Returning for a moment to the general quadratic $\boldsymbol{w} = \boldsymbol{z}^2 + \boldsymbol{cz} + \boldsymbol{d}$, altering the other coefficient of $\boldsymbol{z}$ does not add anything essentially new. When $\boldsymbol{c} = 0$ the two roots are always symmetrically placed with respect to the origin. In the general case the roots can be anywhere. For example, if the roots are at $\boldsymbol{a}$ and $\boldsymbol{b}$, ($\boldsymbol{a}$ and $\boldsymbol{b}$ being complex numbers of course) then

$$\boldsymbol{w} = (\boldsymbol{z} - \boldsymbol{a})(\boldsymbol{z} - \boldsymbol{b}) = \boldsymbol{z}^2 - (\boldsymbol{a} + \boldsymbol{b})\boldsymbol{z} + \boldsymbol{ab}.$$

Now that we understand the quadratic function, it does not take too much intelligence to predict what the general cubic would look like. Here are some plots for the function $\boldsymbol{w} = \boldsymbol{z}^3 - 1$.

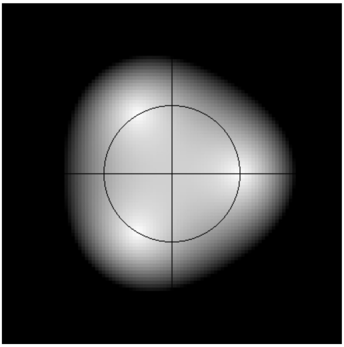


The real part of $\boldsymbol{w} = \boldsymbol{z}^3 - 1$

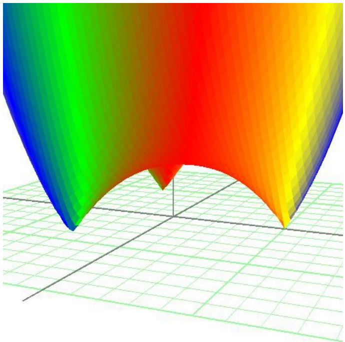


The imaginary part of $\boldsymbol{w} = \boldsymbol{z}^3 - 1$

The lines where the real part is zero obey the relation $x(x^2 - 3y^2) = 1$ and the imaginary ones $(3x^2 - y^2)y = 0$.



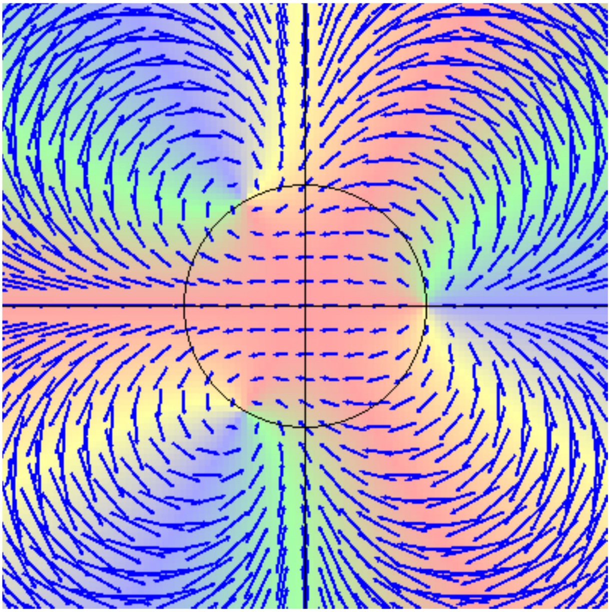
The modulus of $\boldsymbol{w} = z^3 - 1$



The 3D plot of of $\boldsymbol{w} = z^3 - 1$

The three complex roots of unity are clear to see in the modulus plots.

The vector plot is, as always, particularly interesting:



The vector plot of $\boldsymbol{w} = z^3 - 1$

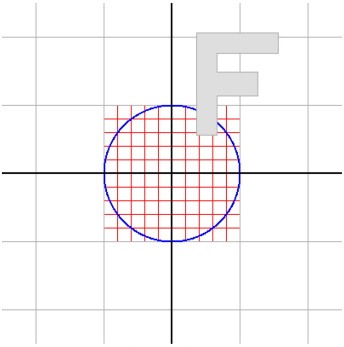
If you navigate your plotting compass round all three roots, you will not be surprised to find that it rotates three times. The winding number of this loop is therefore three.

Nor will you be surprised to find that the winding number of the loop which encloses the root (1, 0) is one. In fact, the winding numbers of the other two roots are also one but there is something different about these loops. Instead of the plotting compass pointing directly away from or directly towards the root, it seems to point around the the root. It seems as if the (1, 0) root is a kind of *source* but the imaginary roots are more like *whirlpools*. We shall discuss this again but for now let us explore another class of functions.

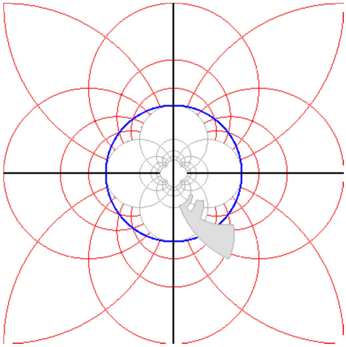
Reciprocal Functions

Now lets turn to a really fascinating function – the reciprocal function $\boldsymbol{w} = 1/\boldsymbol{z}$. This turns a complex number $r\angle\theta$ into $1/r\angle{-\theta}$, that is to say the modulus is reciprocated and the argument negated.

First the mapping plots.

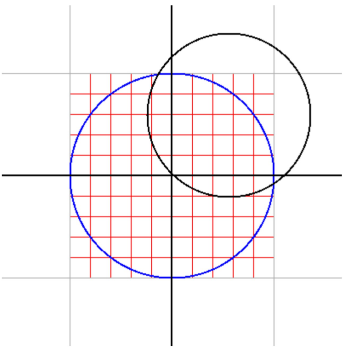


The $\boldsymbol{z}$ *plane*

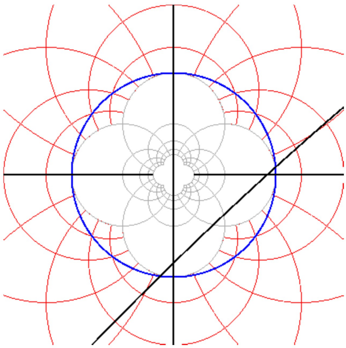


The $\boldsymbol{w}$ *plane* $\boldsymbol{w} = 1/z$

Notice how the whole of the $\boldsymbol{z}$ plane outside the unit circle is mapped onto the $\boldsymbol{w}$ plane inside the circle and vice versa. Notice too how all the straight lines are mapped into circles and how all the circles intersect at right angles.



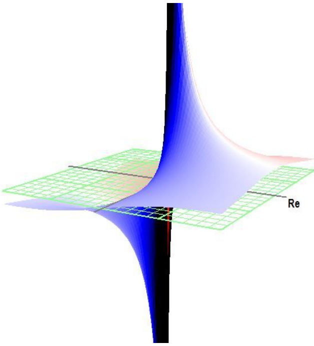
The $\boldsymbol{z}$ *plane*



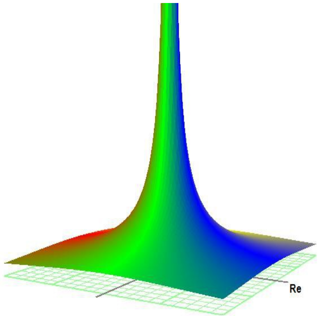
The $\boldsymbol{w}$ *plane* $\boldsymbol{w} = 1/z$

It is also notable that circles are mapped onto circles and that any circle passing through the origin is mapped onto a straight line.

Obviously the point (0, 0) is special in that it is not mapped onto any finite point in the $\boldsymbol{w}$ plane as is clear in the following 3D plots. Points like this are called *singularities*.



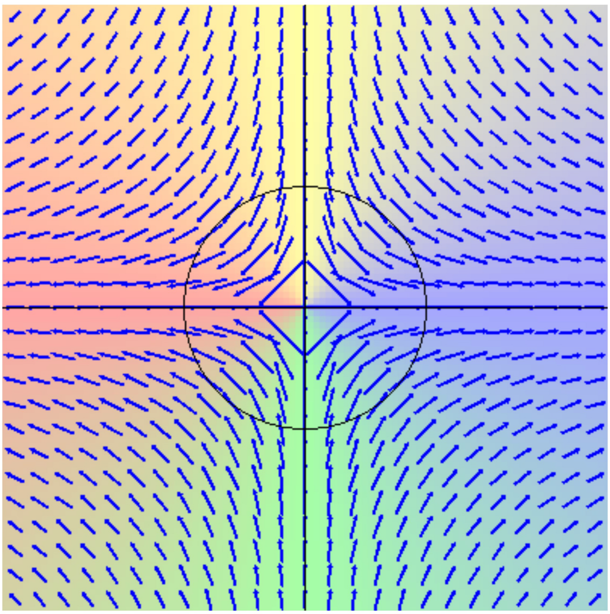
The real part of $\boldsymbol{w} = 1/z$



The modulus of $\boldsymbol{w} = 1/z$

The modulus is always positive so the equation $z^2 = 0$ has no finite roots.

Here is the vector plot.

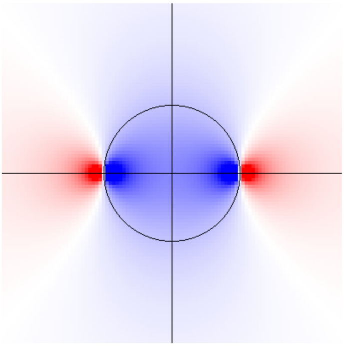


The vector plot of $\boldsymbol{w} = 1/z$

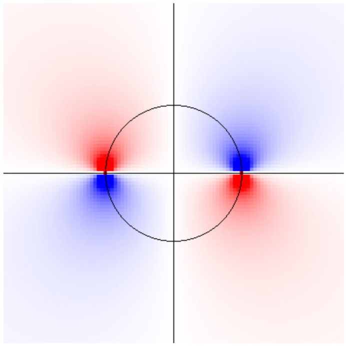
Now if you take your plotting compass for an imaginary excursion round the origin, you will find that it rotates once in the *opposite* direction (and the order of the colours through which you pass is reversed). The *winding number* is therefore –1. This makes sense because the function can be written $\boldsymbol{w} = z^{-1}$. Note that it is not just the roots of a function which are exceptional points; so are the singularities.

How would you describe the behaviour of the flow in the region of the singularity? Mathematicians refer to it as a '*saddle point*' but I prefer to call it a '*collision point*'.

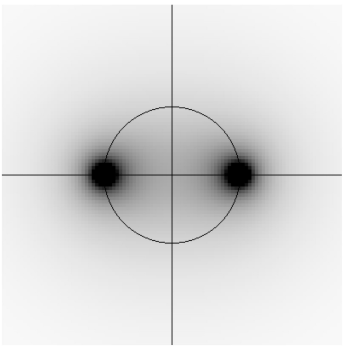
Now lets look at another reciprocal function which has some interesting properties – $\boldsymbol{w} = 1/(z^2 - 1)$. This is what its vital plots look like:



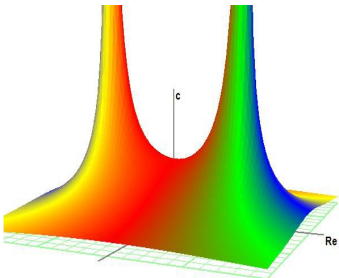
The real part of $\boldsymbol{w} = 1/(z^2 - 1)$



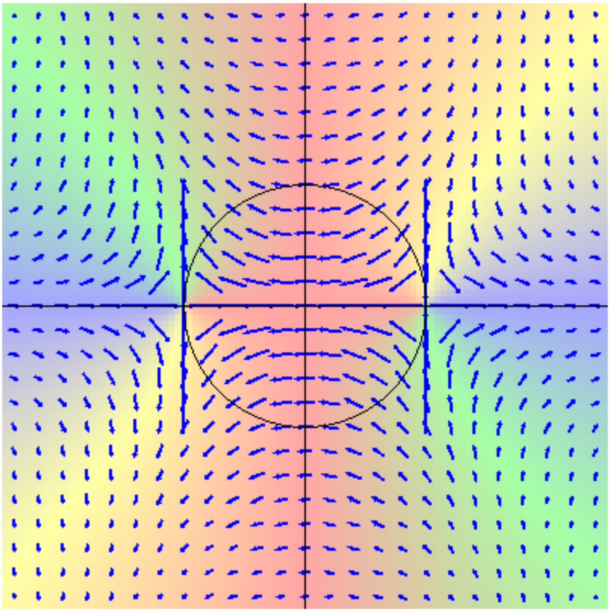
The imaginary part of $\boldsymbol{w} = 1/(z^2 - 1)$



The modulus of $\boldsymbol{w} = 1/(z^2 - 1)$



The modulus of $\boldsymbol{w} = 1/(z^2 - 1)$



The vector plot of $\mathbf{w} = 1/(z^2 - 1)$

Obviously there are two singularities and these points each have winding number –1. The winding number of the function as a whole is therefore –2. Both the singularities are collision points (saddle points).

(You can see why I don't like the term 'saddle point'. Look at the 3D plot of the modulus. There is as clear an example of a 'saddle point' as you could wish for – but it is at the origin, not at either of the singularities!)

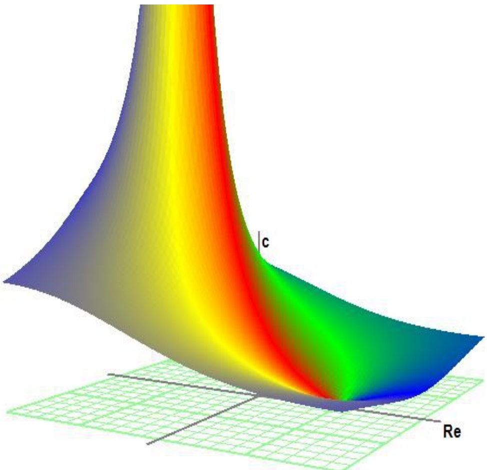
Sources, Sinks and Whirlpools

The rational quadratic function is defined as follows:

$$f(z) = \frac{z + a}{z + b}$$

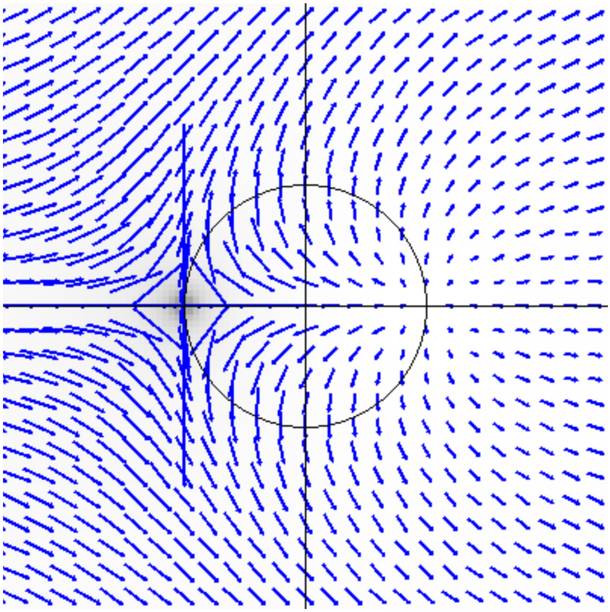
and its general features can be illustrated using **a** = –1 and **b** = +1.

Looking at the modulus plot, what would you predict its winding number was?



The modulus of $\boldsymbol{w} = (z - 1)/(z + 1)$

You can easily see that it has a root at (1, 0) and a singularity at (–1, 0). These will have winding number +1 and –1 respectively so the overall winding number will be zero – a fact which you can check by inspecting the following vector plot (in which the vectors are superimposed on the modulus plot so that you can clearly distinguish the singularity on the left from the root on the right).



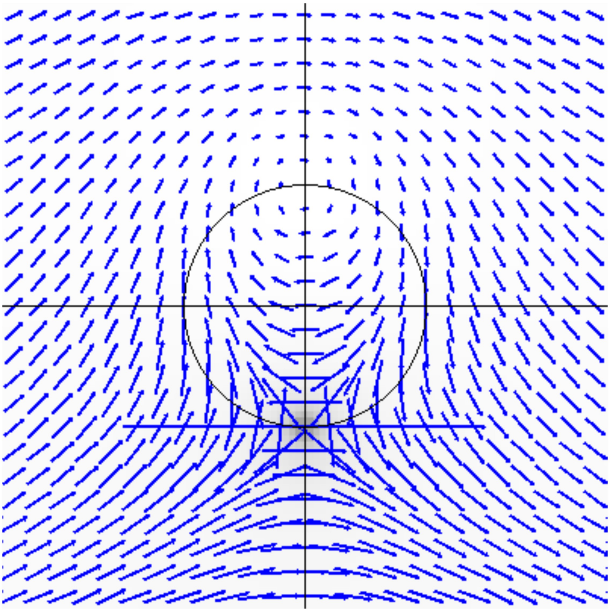
The vector plot of $\boldsymbol{w} = (z - 1)/(\boldsymbol{z} + 1)$

You can also confirm that the root is a source and the singularity is a collision point.

Now in the case of the function $\boldsymbol{w} = \boldsymbol{z}^3 - 1$, we found that two of the roots were whirlpools. This raises two interesting questions. Firstly, how can we determine in advance whether a root is going to be a source, a sink or a whirlpool? and secondly, why are there three different types of root points but, apparently, only one type of collision point?

Well, appearances are deceptive. The human eye is quick to distinguish between a source and a whirlpool but although there are just as many different types of collision point, the difference is a little more subtle.

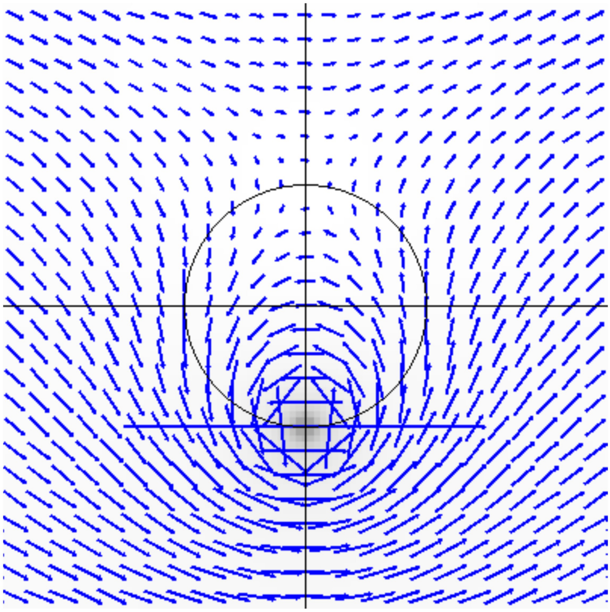
Have a look at the vector plot of the function $\boldsymbol{w} = (\boldsymbol{z} - \boldsymbol{i})/(\boldsymbol{z} + \boldsymbol{i})$ which has a root at $(0, \boldsymbol{i})$ and a singularity at $(0, -\boldsymbol{i})$.



The vector plot of $\boldsymbol{w} = (z - \boldsymbol{i})/(z + \boldsymbol{i})$

The root is obviously a pure whirlpool and the singularity is a collision point. But there is a difference between this collision point and those we have met before. Every collision point we have met so far has a pair of lines which we might call the lines of influx and efflux. These lines are always at right angles. When the singularity lies on the real axis, the influx and efflux lines have always been parallel to the axes; but in the above case the lines are at 45° to the axes. In general the influx and efflux lines can be at any angle depending on the precise function in question and any collision point is characterised by the angle which the lines of efflux make with the real axis.

In fact it is possible to turn any source, sink or whirlpool into an equivalent collision point by using what is known as a Pólya plot of the function. This is exactly the same as the vector plot except that we plot the *complex conjugate* vector. (This is the vector whose angle has been negated.) A source is turned into a collision point whose characteristic angle is 0°; a sink into one whose angle is 90° and clockwise and anti-clockwise whirlpools into +45° and -45° collision points respectively.

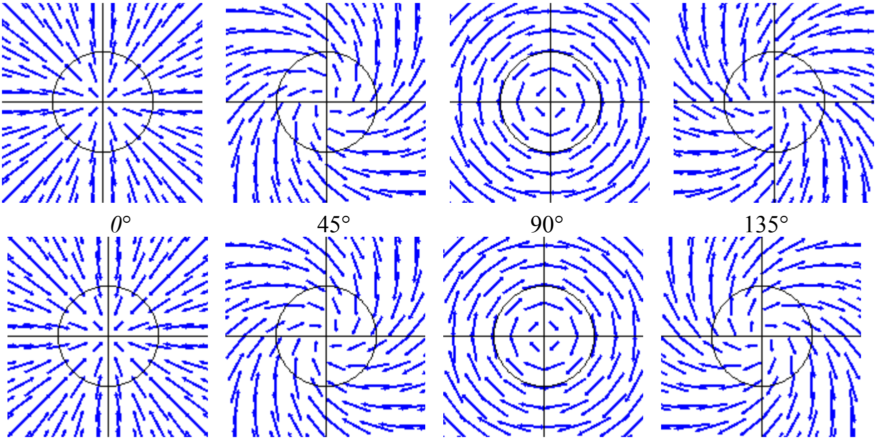


The Pólya plot of $\boldsymbol{w} = (z - \boldsymbol{i})/(\boldsymbol{z} + \boldsymbol{i})$

Note how the collision point has been turned into a whirlpool and the whirlpool into a 45° collision point.

We can summarise our findings so far by categorising all the exceptional points we have met using just two numbers: a) the *winding number* N_w and b) the *initial angle* ψ. The latter is defined as the angle which a vector close to the point makes with the positive real axis. Using this scheme, an exceptional point with a winding number of +1 and an initial angle of zero is a *source*. Increase the initial angle to 90° and we get an anticlockwise whirlpool. With $\psi = 180°$ we have a sink and if $\psi = 270°$ (or – 90°) we have a clockwise whirlpool.

The following series of vector plots show the root of the function $\boldsymbol{w} = \boldsymbol{cz}$ as $\boldsymbol{c}$ moves round the unit circle:

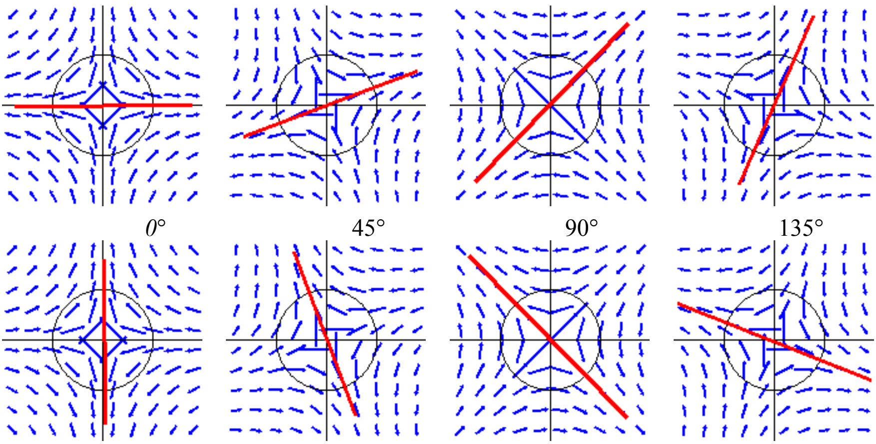


0° 45° 90° 135°

180° 225° 270° 315°

In a similar way, a collision point has winding number –1. If $\psi = 0°$ then the lines of efflux lie along the real axis. As ψ is increased the efflux line rotates at half the speed so that when $\psi = 90°$ the efflux line is at 45° to the real axis. When ψ has rotated all the way round, the efflux lines will be back where they were.

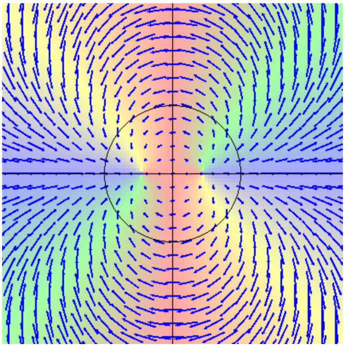
The following series of vector plots show the singularity of the function $\boldsymbol{w} = \boldsymbol{c}/\boldsymbol{z}$ as $\boldsymbol{c}$ moves round the unit circle:



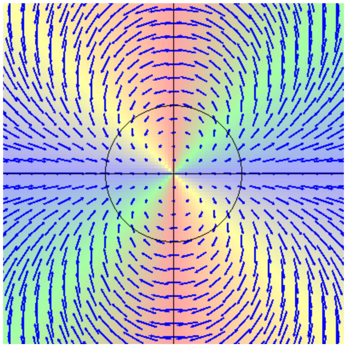
0° 45° 90° 135°

180° 225° 270° 315°

A typical quadratic equation has two roots, each with winding number 1. But there are degenerate cases where the two roots coalesce into one with a winding number of 2. Obviously this cannot be a source, or a sink – or even a whirlpool. So what sort of point is it?



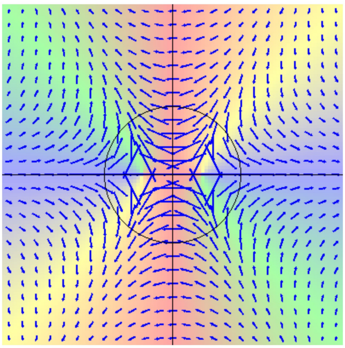
The vector plot of $\boldsymbol{w} = z^2 - 0.2$



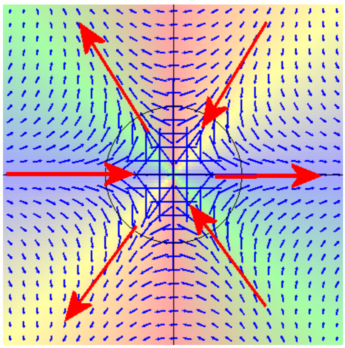
The vector plot of $\boldsymbol{w} = z^2$

The best name I can come up with is a dipole point. (The flow lines have the same sort of look as the field lines round a bar magnet or a pair of equal and opposite electric charges but you should note that the magnitudes are all wrong.) The name also happily suggests the winding number.)

The same sort of thing happens with singularities. Take a look at the vector plots of $\boldsymbol{w} = 1/(\boldsymbol{z}^2 - 0.5)$ and $\boldsymbol{w} = 1/\boldsymbol{z}^2$.



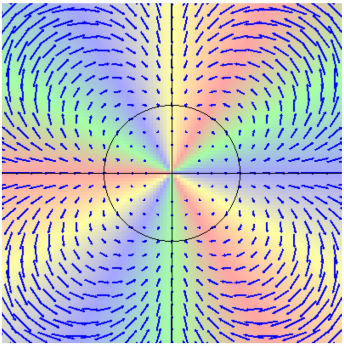
The vector plot of $\boldsymbol{w} = 1/(z^2 - 0.2)$



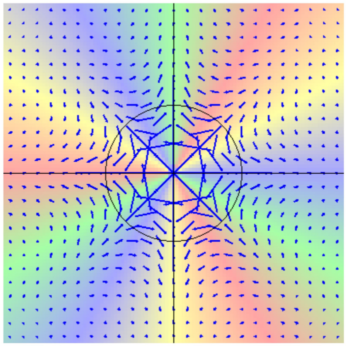
The vector plot of $\boldsymbol{w} = 1/z^2$

As we reduce the constant in the denominator, the two singularities, each with 4 lines of influx and efflux merge into one singularity with 6 lines! Lets call this a di-collision.

Similarly if we coalesce three roots, each with winding number +1, we will obtain what we might call a 'tripole' with winding number +3; and three collision points will make a 'tri-collision' with winding number -3.



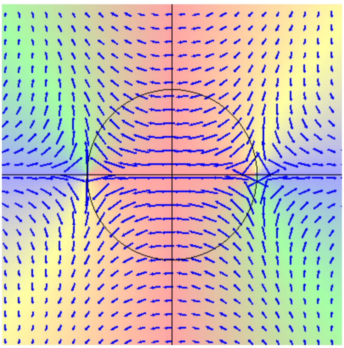
The vector plot of $\boldsymbol{w} = z^3$



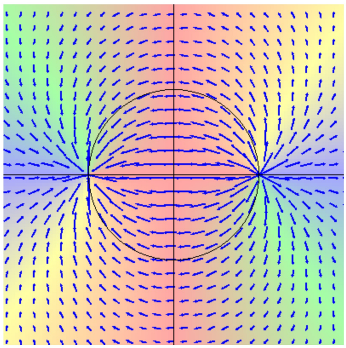
The vector plot of $\boldsymbol{w} = 1/z^3$

Here are four more rather interesting plots, each with the associated Pólya plot.

The first is the reciprocal of $(z^2 - 1)$:



The vector plot of $\boldsymbol{w} = 1/(z^2 - 1)$



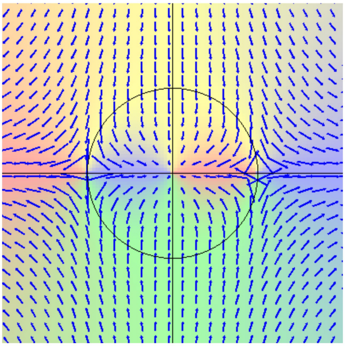
The Pólya plot of $\mathbf{w} = 1/(z^2 - 1)$

It has singularities at ±1 and, as you can see, these are both standard collision points with winding number ±1 one of which has initial angle 0° and the other 180°.

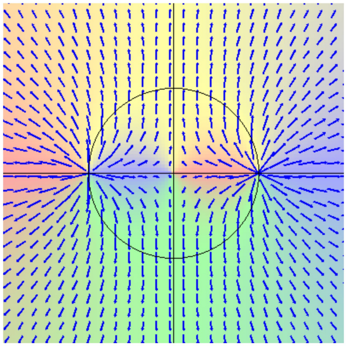
The Pólya plot (which is the complex conjugate of the vector plot) consists of a source and a sink.

The reason why the Pólya plot is particularly interesting is that, because the function is a reciprocal function, the magnitude of the vectors gets smaller as you move further away from the sources. This plot looks a lot more like the magnetic field round a bar magnet or the electric field near a pair of equal and opposite charges.[2]

If we replace the numerator with z this is what we get:



The vector plot of $\mathbf{w} = z/(\mathbf{z}^2 - 1)$

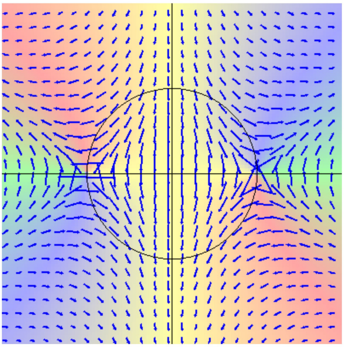


The Pólya plot of $\mathbf{w} = z/(z^2 - 1)$

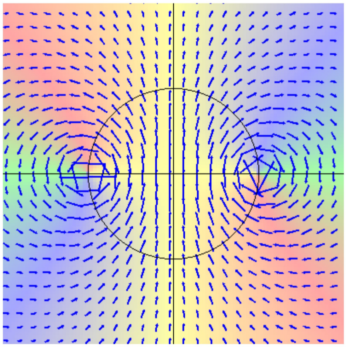
We still have the two singularities at ±1 (but this time with the *same* initial angle) but a root has appeared at $z = 0$. Now instead of having a source and a sink, the Pólya plot shows two sources and a collision point in between.

Next we shall replace the numerator with **i**:

2 It isn't exactly right because these functions are based on a simple reciprocal law while magnets and charges obey an inverse *square* law. The electric field near a long straight wire does, however, obey a simple reciprocal law so these fields could represent the fields near an electricity cable, for example.



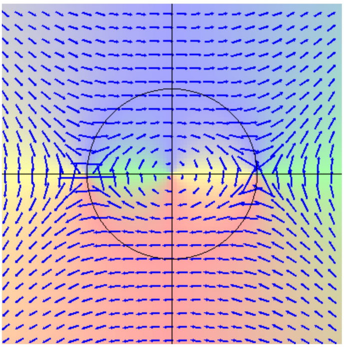
The vector plot of $\mathbf{w} = \mathbf{i}/(\mathbf{z}^2 - 1)$



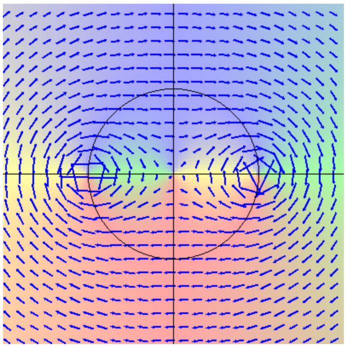
The Pólya plot of $\mathbf{w} = \mathbf{i}/(z^2 - 1)$

The effect of this is to rotate all the vectors by 90°. As we have seen (page 24), this rotates the lines of influx and efflux by 45° so that the lines in the vector plot no longer join hand in hand, as it were; they merge asymptotically. Likewise on the Pólya plot, the two singularities are turned into whirlpools which rotate in the opposite direction. This makes the field resemble that round a pair of equal charges.

Finally, to complete the set we put the numerator equal to $\mathbf{i}z$:



The vector plot of $\mathbf{w} = \mathbf{i}z/(\mathbf{z}^2 - 1)$

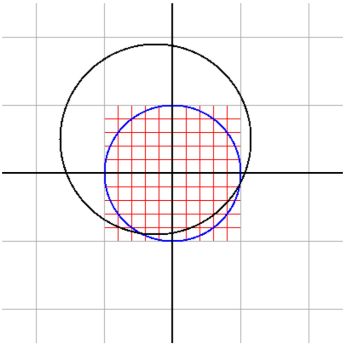


The Pólya plot of $\mathbf{w} = \mathbf{i}z/(z^2 - 1)$

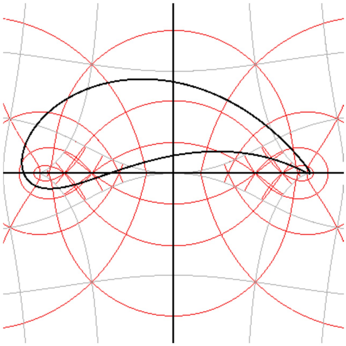
As before this introduces a root at the origin and the Pólya plot shows the effect produced when two clockwise whirlpools exist side by side.

It should be clear from these examples that the study of complex functions has important applications in many disciplines in applied science such as electromagnetism and fluid dynamics.

As a final example which is of particular interest to aeronautical designers we shall briefly consider the function $\boldsymbol{w} = \boldsymbol{z} + 1/\boldsymbol{z}$ or, if you prefer, $\boldsymbol{w} = (\boldsymbol{z}^2 + 1)/\boldsymbol{z}$. Here is its map.



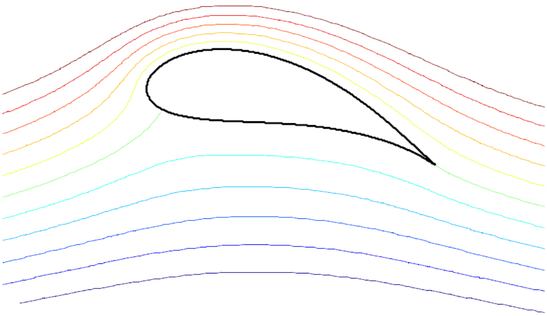
The z plane



The map of $\boldsymbol{w} = (z^2 + 1)/z$

Notice how the circle has been transformed into a quite convincing aerofoil shape.

Now the flow lines of an airstream flowing past a circle are fairly easy to work out from first principles and it turns out that this transform correctly transforms the flow lines as well thus giving designers a simple way of calculating the flow of air over a wide variety of possible wing shapes.



The flow of air over a simple aerofoil as modelled by a complex function

Complex Differentiation

Perhaps the most obvious feature of a conventional graph of a real function is the fact that the gradient of the function varies from place to place and that, where the function reaches a maximum or a minimum, the gradient is necessarily zero.

Complex functions do not have an obvious 'gradient', nor do they have 'maxima' and 'minima'. But we can extend these concepts into the complex domain by applying the same rules and processes which we use to define the gradient of a real function to a complex one.

To measure the gradient of a real function we measure how much the function changes by when we increase the variable x by a small amount. We then divide the change by the increase. More formally, the gradient of a real function is defined as the limit of

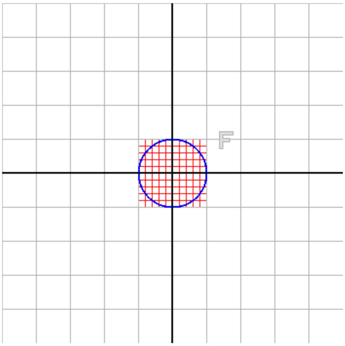
$$\frac{f(x + \delta x) - f(x)}{\delta x}$$

as δx is made smaller and smaller.

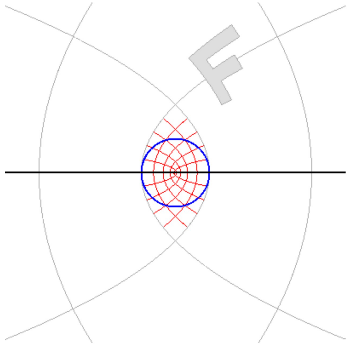
In the same way, we define the 'gradient' of a complex function by increasing the variable z by a small amount, measuring the change in the function and the dividing the latter by the former.

But we have a problem here. In what direction should we 'increase' the variable z? z is a sort of vector and we could, in principle, 'increase' it in any direction. Fortunately, however, we have a lucky escape. It turns out that all the functions that we have looked at so far (and all those that we shall consider later) have a remarkable property. *It doesn't matter which direction you choose to move z, the 'gradient' will always work out to be the same*! (Mappings which have this important property are called *analytic functions*.)

One other thing. We should stop calling the quantity we have calculated the 'gradient' because it is, of course, a complex number not a real one. Tristan Needham has coined an excellent word for this quantity: he calls it an *amplitwist*. The reason for this is as follows. Because of the remarkable property of an analytic mapping, if you consider what happens to a tiny figure like a letter F, you will discover that, as you move it around the z plane, its image in the w plane remains unaltered; the only thing that happens to it is that it is either expanded (or shrunk) and it is rotated. If the expansion factor is A and the angle of rotation is α then the amplitwist at that point in the z plane is the complex number $A\angle\alpha$. For an example, consider the function $w = z^2$.



The z plane



The map of $w = z^2$.

The letter F is positioned at the point (1.5, 1) in the z plane. When mapped by the square function it remains recognisably a letter F but it has been expanded by what looks like a factor of about 4 and rotated by about 30°. In other words, the amplitwist of the z^2 function at the point $1.5 + i$ is $4\angle 30°$ – or, if you prefer, $3.5 + 2i$.

Obviously this is only an approximate guess – but the principle is the important thing: it is possible to assign a new complex number to every point in the z plane which effectively tells us how the function is changing in the region of that point, just as the gradient of a real function tells us how the function changes in that region.

The process of calculating the amplitwist of a function is, of course, called differentiation and, amazingly, you can use exactly the same rules to differentiate a complex function as you use with a real one. In particular, the amplitwist of z^2 is $2z$. (and at the point $1.5 + i$ it is exactly $3 + 2i$ so our approximate guess was not too far out after all.)

Now we have figured out how to differentiate a complex function, it is of great interest to investigate those special places where the amplitwist is zero. Consider the function $w = z^2 - 1$. This has two roots at (1, 0) and -1, 0) which we have noted are exceptional points with winding number +1.

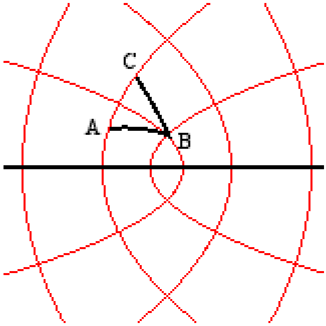
The amplitwist of this function is, of course, $2z$ and this is zero in just one place – the origin.

Now what, exactly does it mean to say that the amplitwist is zero? It means that the amplitwist vector is $0 + 0\boldsymbol{i}$ or in polar coordinates $0\angle$???. The problem is clear. If the vector has zero magnitude then its direction is indeterminate. A tiny letter F placed at this point would be shrunk down to such a small size that it is impossible to say what its orientation is. Obviously, places where the amplitwist is zero are much more significant than either the roots of the function or its singularities; they are places where the whole *analytic* nature of the function breaks down. Everywhere else, the function behaves predictably, if bizarrely, but here, anything goes. They are like the centre of a black hole – where the whole of a region has been compressed to an infinitesimal point.

It is with good reason, therefore that places where the amplitwist is zero are called *critical* points.

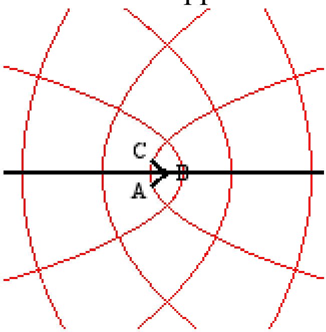
But what exactly do we mean when we say that the 'analytic nature of the function breaks down' at a critical point? To answer this question we need to ask ourselves what exactly is the property that makes some functions 'analytic'. So far, all we have said is that analytic functions are those functions which have the same amplitwist, whatever direction you measure it in. Now consider the effect of an analytic function on a small letter V whose apex B is placed at the point in question and whose arms A and C make an angle θ at the apex. Now what we are saying is that, however the V is oriented, an analytic mapping will always amplify and twist it by the same amount. In other words, the angle θ will *remain the same*. You can see what I mean if you look at the transformation of the letter F on the previous page. The letter is amplified, twisted (and slightly distorted) but all the angles between the lines are exactly 90°. Mappings which preserve angles like this are called *conformal* mappings and all analytic mappings are conformal. (It is also true to say that all conformal mappings are analytic but I shall not bother to prove this.)

So now we possess a powerful tool to test whether or not a given mapping is analytic or not. We let our test letter V wander over the mapping and see whether the angle θ stays the same. Let's test the function $w = z^2 - 1$. All seems to be well; the angle is unchanged; but when we bring the test letter close to the origin we begin to run into problems. Firstly, the image of our letter gets rather small so we have to magnify our plot of the w plane and when we do this we see that the V has become quite distorted like this:



However, careful measurement of the angle at the apex reveals that the angle is unchanged. On the other hand, we have to note that the angle between the (straight) line AB and the (straight) line BC is rather larger than the angle at the apex.

Now we take the plunge and place B exactly on the origin – the critical point of this function. What happens now?



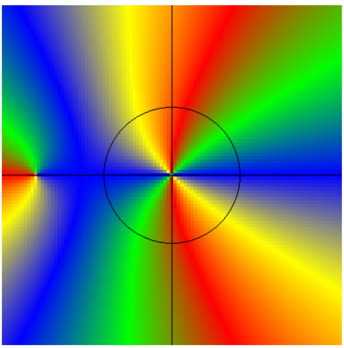
The letter has shrunk so much we can barely see it but it is just big enough to measure the angle ABC which turns out to be 2θ. This should be no surprise as the whole point of the quadratic function is that it doubles the argument of rays extending from the origin. But what has happened to the angle at the apex B? Is it θ or is it 2θ?

On page 205 of Needham's excellent book he says of this situation 'When the z^2 mapping acts on a pair of rays through the critical point $z = 0$, it fails to preserve the angle between them ; in fact it doubles it.'

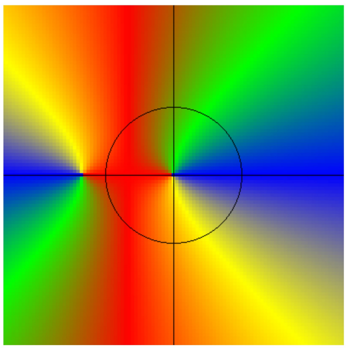
Now I find this statement either false or at least, very misleading. It seem to me to be unreasonable to claim that the function is conformal everywhere *except at the critical point*. Yes the critical point is special – but it is not that special. It is not the case that as you move the apex B of our test letter along a line which passes through the critical point the angle at the apex suddenly doubles at that point. The truth of the matter is – you can't measure the angle at the apex because the test object has shrunk to zero size. It is better to say that at the critical point the angle becomes indeterminate.

Notwithstanding my silly quibbles over phraseology, it is clear that something special does indeed happen at the critical point where the amplitwist is zero.

Consider now the function $\boldsymbol{w} = z^3 + 2z^2$. This has a double root at $z = 0$ and a third root at $z = -2$. Using the standard methods for differentiating a function we find that it has critical points at the origin and at $-4/3$. All these features are easily visible on the argument maps below:



The argument map of $\boldsymbol{w} = z^3 + 2z^2$



The amplitwist map of $\boldsymbol{w} = z^3 + 2z^2$

It is of interest to note that the winding numbers of the two roots are +2 (at the origin) and +1 and that the winding numbers of the critical points are both +1. In general, the total winding number of the amplitwist function will be one less that the total winding number of the original function because the act of differentiation reduces the power of a polynomial by one.

If the winding number of a critical point is greater than 1, it means that this point is 'even more critical' than usual!

It might be worth briefly mentioning here something about non-analytic functions. We have shown that the mapping $\boldsymbol{w} = \boldsymbol{z}^2$ is conformal and hence analytic. Often, for computing purposes, we break this formula down into its real and imaginary parts i.e.

$$u = (x^2 - y^2)$$
$$v = 2xy$$

But there is no reason why we should stick to this pair of functions. Any functions will provide us with a mapping of some kind. So what makes some mappings conformal and others not?

Well it is pretty easy to see that if two functions $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ are conformal, then their sum $f(\boldsymbol{z}) + g(\boldsymbol{z})$ will also be conformal. It is also true to say (but a bit more difficult to prove) that the product (and quotient) will also be conformal, as is $\boldsymbol{c}\, f(\boldsymbol{z})$ where $\boldsymbol{c}$ is a complex constant.

Now it is obvious that the function $f(\boldsymbol{z}) = \boldsymbol{z}$ is conformal because this mapping doesn't change anything at all. This means that $\boldsymbol{cz}$ is also conformal; so is $\boldsymbol{cz}^2 + \boldsymbol{z}$; indeed so is any polynomial function in $\boldsymbol{z}$. Thus all the functions we have so far considered are conformal and hence analytic.

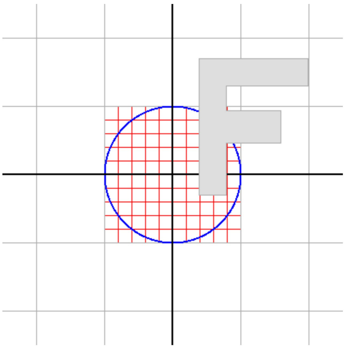
But an arbitrary mapping will not, in general be conformal; it will not have a consistent amplitwist at every point; it will not be differentiable; it will not be analytic. In short, it will not be nearly so interesting.

(This also gives us a clue as to why some mappings generate consistent fractals like the Mandelbrot map while others result in an unattractive jumble.)

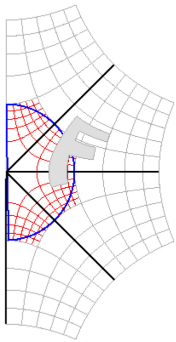
Multivalued Functions

All the polynomial functions we have considered so far have been *rational* functions – i.e. functions involving only integral powers of z. We need to consider now what happens if we permit the use of fractional powers of z like the square root etc. In particular, we shall be interested to find out what happens to the concept of winding number (which is, by definition, an integer) when applied to a vector map of an irrational function.

Lets start with the map of $w = \sqrt{z}$.



The z plane

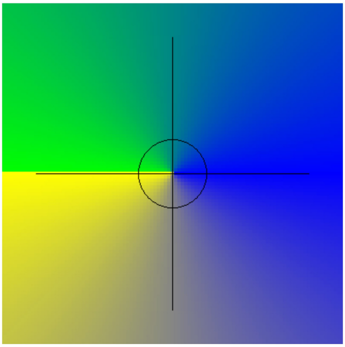


The map of $w = \sqrt{z}$.

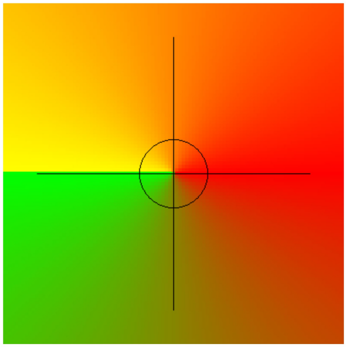
You may remember that the square function squares the modulus and doubles the argument. In the same way, the square root function takes the square root of the modulus and *halves* the argument. This is why the whole of the z plane is mapped onto half of the w plane.

Except, of course, that we have not taken into account the fact that the square root function has two possible values. Here we have only plotted the positive value. To complete the picture imagine the positive half rotated about the origin by 180°.

Now lets have a look at the argument of the function:



The argument of the positive root

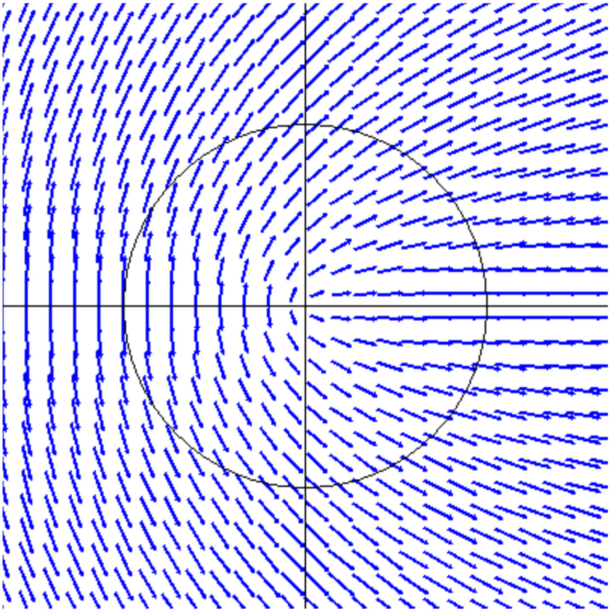


The argument of the negative root

Take a walk round the unit circle starting from (1, 0) where the argument is zero (blue in the left hand map). As you cross the imaginary axis the argument increases to 45° and enters the green region. But when we reach (-1, 0) we have to jump to the right hand map (i.e. the negative root) to continue our journey smoothly and when we reach (1, 0) again the argument has only got to 180°. We need to make another complete circuit before the argument reaches 0 again.

To make a model of this, cut out two circular pieces of paper A and B. Make a single radial cut in each piece. Now place A on top of B and sellotape one of the cut edges of A to the opposite cut edge of B. Now the tricky bit! Sellotape the other cut edge of A to the other cut edge of B. I know this can't be done in ordinary 3D space but it can be done in 4D!

It is clear that the origin is a very strange sort of exceptional point. It is technically called a *branch point*. Lets try to determine its winding number. Here is the vector map (of the positive function).



The vector plot of $\boldsymbol{w} = \sqrt{z}$

Look carefully at the negative real axis. You will see that the arrows on each side point in opposite directions. This indicates that there is a discontinuity here[3]. As we have seen, in order to make the argument rotate once, we have to make two circuits of the branch point. Alternatively, in one circuit of the branch point, the argument rotates through half a revolution. Either way, the winding number of this point is ½.

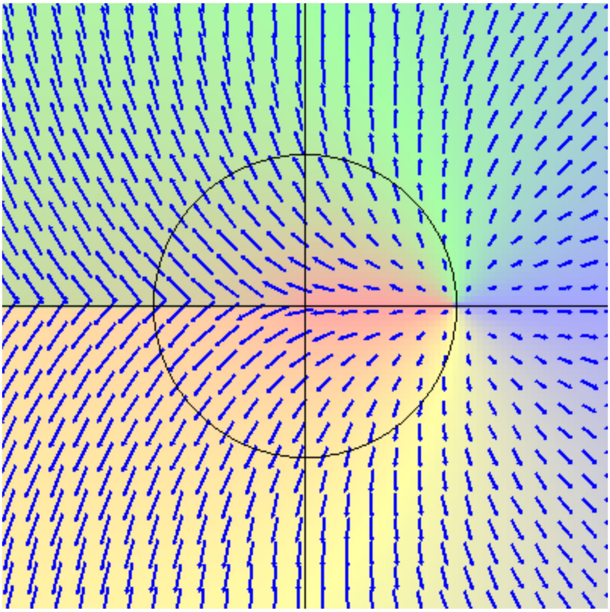
Obviously the pole of the function $\boldsymbol{w} = \sqrt[3]{z}$ will have winding number 1/3 and you would need three discs of paper to model it.

If the exponent was 2/3, the winding number would also be 2/3. What this means is that in 3 circuits of the pole the vector would rotate twice.

3 Actually there is nothing special about the negative real axis. It only appears so because I have chosen to use only the principal arguments from $-\pi$ to $+\pi$. You could, in principle make any cut you like as long as it terminates on the branch point.

And if the exponent was itself irrational (e.g. $1/\sqrt{2}$) then no whole number of circuits of the pole would result in a whole number of rotations of the vector. You would need an infinite pile of discs to model the behaviour of the function!

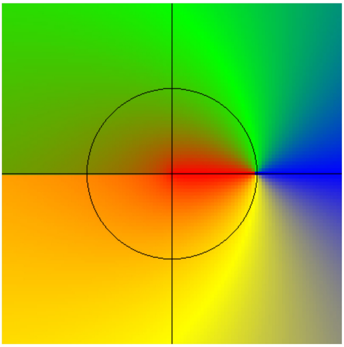
To some extent the picture is confused because the branch point of $\sqrt{z}$ is also a root of the equation. Let's look at the function $\boldsymbol{w} = \sqrt{\boldsymbol{z}} - 1$. The single root is now at +1 but the branch point is still at the origin as the vector plot shows:



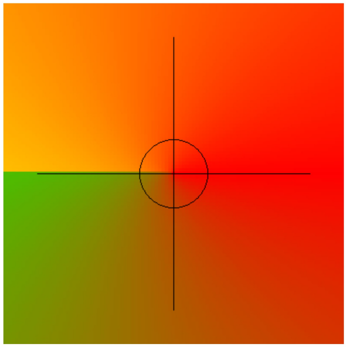
The vector plot of $\boldsymbol{w} = \sqrt{z} - 1$

As before the cut has been made along the negative real axis and the arguments on each side of the cut are different – *but they are not opposite*. Notwithstanding this, if we take a walk round the branch point, when we step from the positive root to the negative root as we cross the branch cut, the argument cannot jump discontinuously from one value to another – there are no discontinuities in this function.

The puzzle is resolved when we look at both discs:



The argument of the positive root



The argument of the negative root

In the first disc on the left (which uses the positive root) the argument rotates from blue (0°) through green (90°) to about 135°. Then we step onto the second (negative root) disc which takes us through red (180°) to about 225°. Finally we step back onto the positive disc to complete the journey through yellow (270°) back to zero. The winding number of this branch point is clearly ½. Even though one of the discs takes us through 3 times as much angle as the other, just two rotations are needed to complete one circuit of the argument.

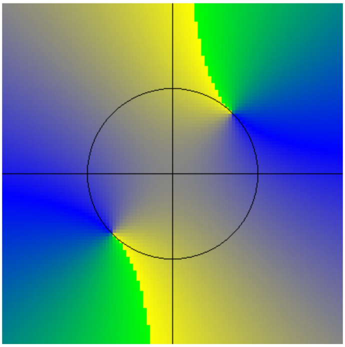
Now we noted earlier that critical points are places where the amplitwist of the function (i.e. the derivative) is zero. What can we say about the amplitwist of a multifunction at its branch point?

Well, the derivative of $\sqrt{z}$ is $\frac{1}{2\sqrt{z}}$ and it has a singularity at $z = 0$. In fact I believe that I am right in saying that the amplitwist of a branch point is always infinite. The converse is not true, however. Consider, for example, the function $w = 1/z$. Its derivative is $-1/z^2$. Obviously both have singularities at $z = 0$, but $1/z$ does not have a branch point because it is not a multifunction.

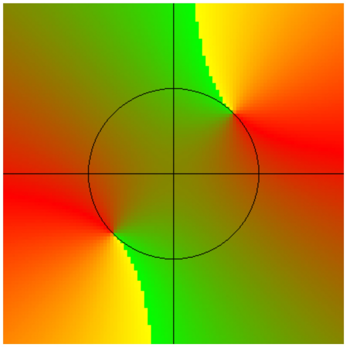
Another interesting function is $w = \sqrt{z^2 - i}$. Its roots are at (1, 1) and -1, -1) and its derivative has singularities at both these points too.

Here are the argument plots of both the function and its derivative.

First the function:

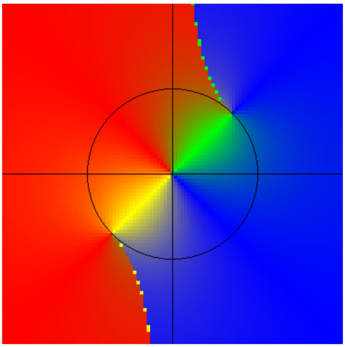


The argument of the positive root

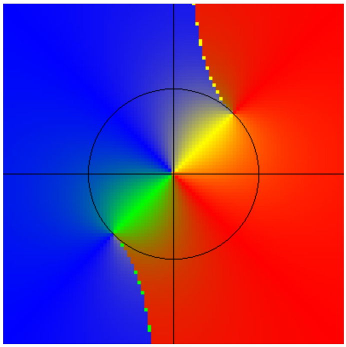


The argument of the negative root

and its derivative:



The argument of the positive root



The argument of the negative root

(Well they are nothing if not colourful!)

To make a model of this function, make two curved cuts in two discs of paper A and B. (Actually the cuts can be anywhere as long as they end on the two branch points. The ones shown in the images are simply the result of the arbitrary decision to restrict the arguments to the principal value.) Now, as before, sellotape A to B in two places – and now take a trip into 4 dimensions and sellotape the other two edges.

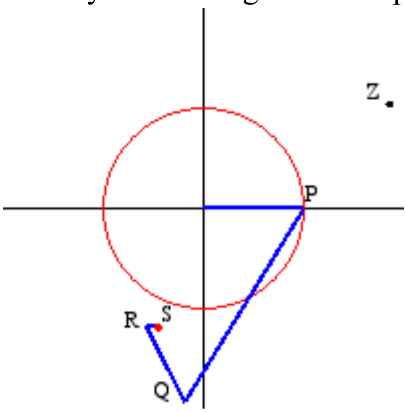
Series Functions

Let us define a new function as follows: (I shall use bold type to distinguish the complex function **cos**(z) from the real function cos(x).)

$$\mathbf{cos}(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \ldots \qquad (1)$$

All the terms are analytic so we can be confident that the whole function is analytic too. We can also be confident that the series will converge for all values of z because the factorial function increases more rapidly than any power.

This is how it works: taking $z \approx 2 + \mathbf{i}$, the first term takes us to P, the point (1, 0), then we subtract ½ z^2 which takes us to Q; the next term takes us to R etc. Eventually we converge onto the point S.



The first few terms of the **cos** *function*

Now lets see if we can predict what the features of this function will look like.

If we restrict ourselves to the real axis – i.e. if we put $z = x$, our function becomes:

$$\mathbf{cos}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \qquad (2)$$

which we recognise as the expansion of cos(x).

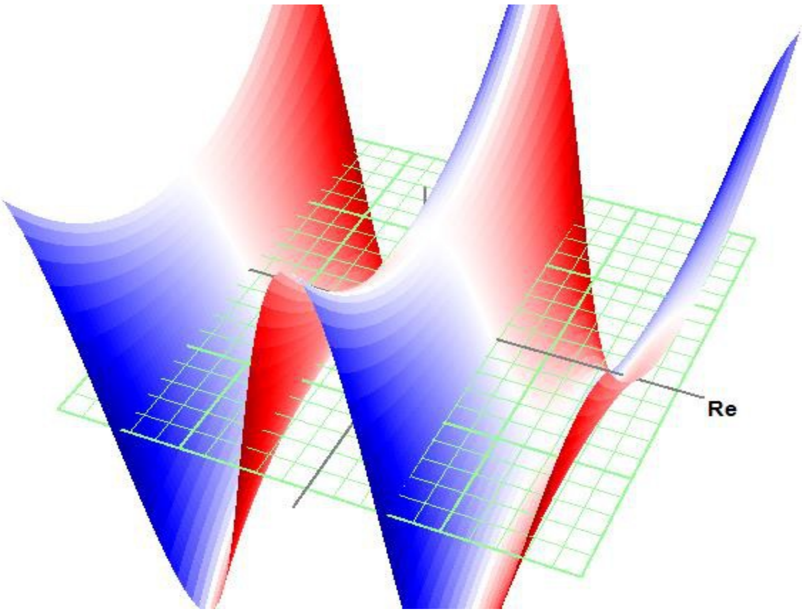
What this means is that a slice through the complex function along the real axis will look like a cosine curve. What is more, all these points are themselves real – that is to say, they have zero imaginary component. This means that the modulus of the function will also look like a cosine curve.

What about a slice along the imaginary axis? These are points where $z = \boldsymbol{i}y$. Substituting this into equation (1) and remembering that $\boldsymbol{i}^2 = -1$ we get:

$$\mathbf{cos}(\boldsymbol{i}\,y) = 1 + \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \frac{1}{6!}y^6 + \ldots \qquad (3)$$

Now this function is none other than the hyperbolic cosine – cosh(y). Once again, we note that all the terms are real so the imaginary component of the function is zero along the imaginary axis too.

The real and imaginary axes are obviously rather special and we cannot hope to predict what the result be in the general case but we can get a computer to do this for us and the results are as follows: First the real component.

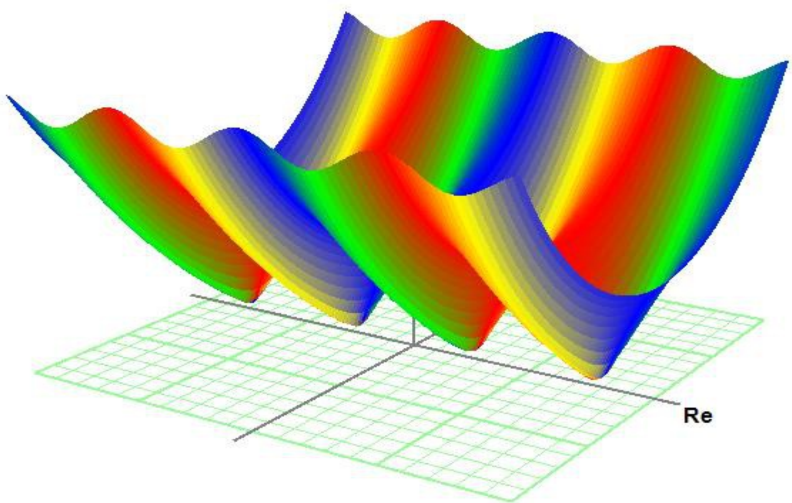


The real component of **cos**(z)

You can clearly see the cosine curve along the real axis and the cosh curve along the imaginary axis. The plot also shows the imaginary component in colour and you can verify that the imaginary component is zero (white) along the axes too.

What comes as a bit of a surprise is that the whole function repeats every 2π along the real axis. This is not at all obvious from equation (1) but the reason will (perhaps) become clear later.

Now the modulus plot:



The modulus of **cos**(z)

See how it dips down to zero at certain points along the real axis (where **cos**(z) = 0). Another interesting thing to note is that the argument (shown in colour) is a series of parallel stripes. What this means is that the argument of the function does not depend on the imaginary component of z, only the real component.

All very mysterious!

The sin function has the following series expansion:

$$\mathbf{sin}(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \ldots \qquad (4)$$

Applying the same arguments as before it is easy to show that the slice along the real axis is a (real) sine curve and a slice along the imaginary axis is an (imaginary) hyperbolic sine (sinh) curve. (That is to say, the values of the function have zero real component and are entirely imaginary.) The only essential difference to the real and modulus plots is that they are shifted by π/2 along the real axis.

Now we shall consider the following function which we shall call **exp**(*z*):

$$\mathbf{exp}(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \ldots \tag{5}$$

Putting $z = x$ it is clear that the slice along the real axis is the familiar exponential curve.

Putting $z = \boldsymbol{i}y$ we get

$$\mathbf{exp}(\boldsymbol{i}\,y) = 1 + \boldsymbol{i}\,y - \frac{1}{2!}y^2 - \boldsymbol{i}\frac{1}{3!}y^3 + \frac{1}{4!}y^4 + \ldots \tag{6}$$

Gathering together the real and imaginary terms we have:

$$\mathbf{exp}(\boldsymbol{i}\,y) = (1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \ldots) + \boldsymbol{i}(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \ldots) \tag{7}$$

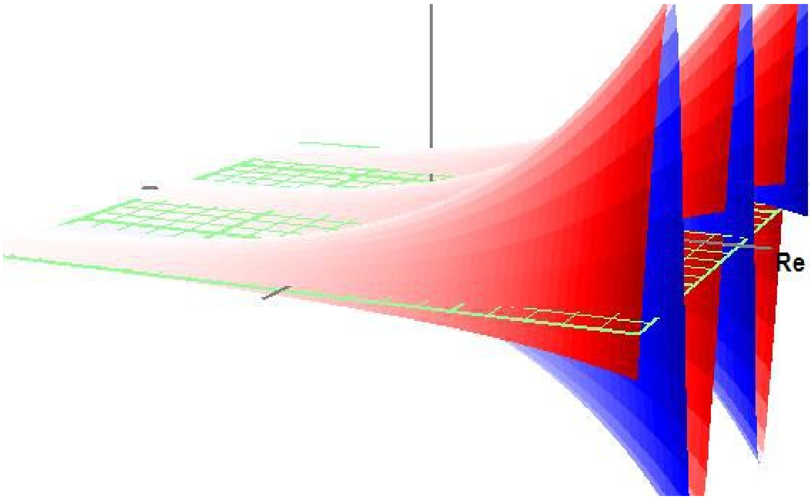
which is obviously means that:

$$\mathbf{exp}(\boldsymbol{i}\,y) = \cos y + \boldsymbol{i}\sin y \tag{8}$$

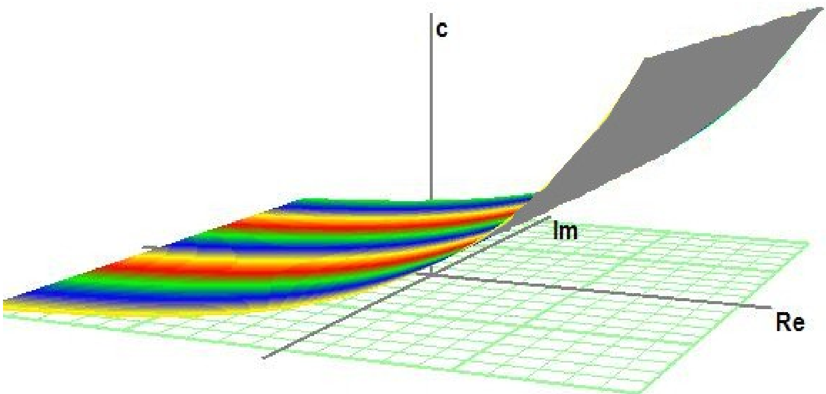
Now the modulus of this number is $\sqrt{(\cos^2 x + \sin^2 y)}$ and its argument is just *y*. So along the imaginary axis. The real component of **exp**(*z*) will be a cosine curve; the modulus will be constant and the argument will be equal to the imaginary component of *z*.

Lets see what this function actually looks like and verify these predictions: The real component is a cosine curve which gets exponentially larger as it sweeps along the real axis. (The imaginary component is a sine curve which does exactly the same.) The modulus looks like a ski jump – a flat sheet bent into an exponential curve while the argument consist of parallel stripes parallel to the real axis.

To put it another way, it appears that the modulus of **exp**(*z*) is e^x and the argument is *y*.



The real component of **exp**(*z*) *with the imaginary component in colour*



The modulus of **exp**(*z*) *with the argument in colour*

This is all very important and very interesting but we haven't really got to the bottom of *why* these functions take these shapes. We may have deduced what the shapes must be along the two axes, but there doesn't seem to be any *a priori* reason why the shapes should remain the same along *other lines*. Why, for example, do *all* the slices through the real part of the **cos**(z) curve parallel to the real axis have the shape of a cosine curve? Why is the modulus of the **exp**(z) curve constant along *all* the lines parallel to the imaginary axis?

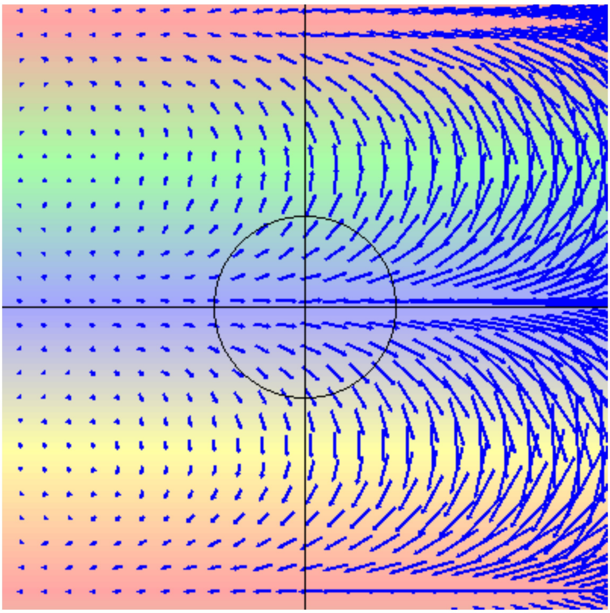
To find the answers to these questions we must look at the derivatives of these functions – i.e. their amplitwist.

First let us recall that it is a *defining* characteristic of the exponential function e^x that it is equal to its own derivative. It is this property that enables us to calculate the coefficients of the terms in the expansion of e^x namely:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots \qquad (9)$$

Since complex polynomials can be differentiated in exactly the same way as real ones, we can confidently say that the amplitwist of **exp**(z) must be **exp**(z) at all values of z.

So lets have a look at the vector map of **exp**(z):

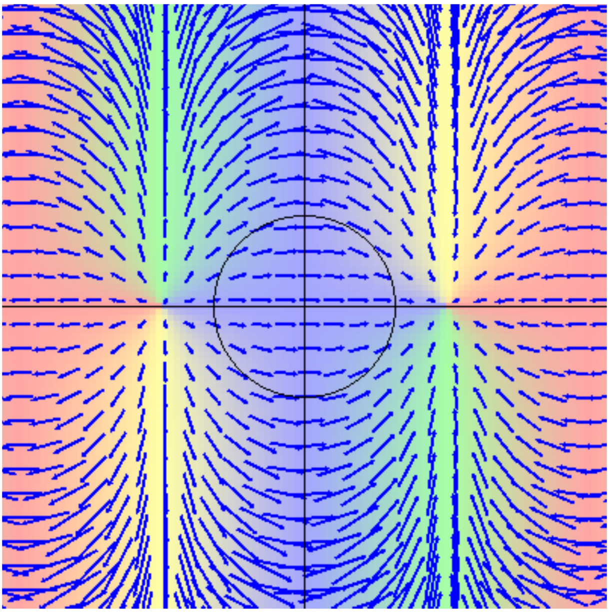


The vector plot of ***w*** *=* ***exp***(z)

You can see that the lines of equal argument are straight and parallel to the real axis. My question, therefore, is this – why are the lines of equal argument parallel to the axis and not, say, rectangular hyperbolae?

Now with an ordinary graph, you can see at once how the gradient of the graph differs from place to place, but it is very difficult, not to say all but impossible, to look at a vector map and immediately deduce the map of its amplitwist. But this vector map is unique because it is *the only vector map where it is everywhere equal to its own amplitwist*. This does not quite answer my question but I am confident that a proof of my statement about the lines of equal argument can be deduced from the italicised sentence above even though I cannot quite see my way to a simple proof at the moment.

Likewise, I am confident that my question about the **cos** function can be answered by starting with the defining characteristic of that function – namely that its second order amplitwist is equal to the negative function – as should be immediately obvious (!) from its vector plot:



The vector plot of ***w*** *=* ***cos(****z****)***

Power Functions

If you have read this book so far, you may have been wondering why I have rigorously avoided using the exponential notation e^{z} where $\boldsymbol{z}$ is a complex number. The answer to this is that any new notation is meaningless until we define its meaning – and it is not at all obvious how we should go about defining what we mean by a quantity e^{z} where $\boldsymbol{z}$ is a complex number.

On the other hand, whatever it means, we require that, by analogy with the real function e^{x}, the derivative of e^{z} must also be e^{z}. Now we have already established that the derivative (or amplitwist) of $\mathbf{exp}(\boldsymbol{z})$ is $\mathbf{exp}(\boldsymbol{z})$ so this basically establishes that $e^{z} = \mathbf{exp}(\boldsymbol{z})$. All we have to do now is verify that e^{z} obeys the usual rules for exponents namely that e^{z} squared is equal to e^{2z} and that e^{z+1} is equal to $e\ e^{z}$ etc. If we take equation (5) as our starting point this is not an easy thing to do and, as this is not a maths text book, I do not propose to do this. Take it from me – the rules do apply and we can manipulate complex exponents as easily as real ones.

Bearing this in mind, how do we calculate the real and imaginary components of e^{z} where $\boldsymbol{z} = x + \boldsymbol{i}y$?

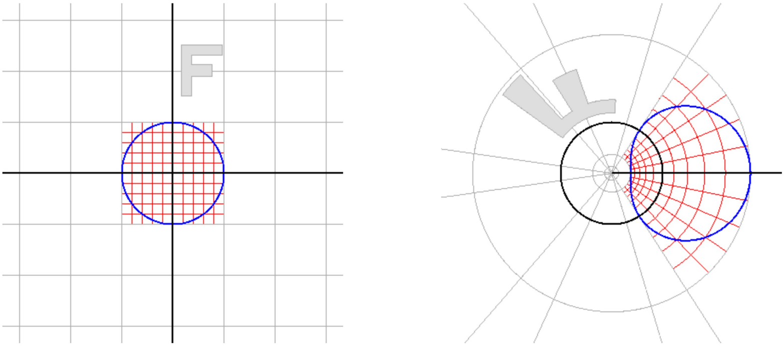
Well, for a start, e^{x+iy} equals $e^{x}.e^{iy}$. Now we have already established that e^{iy} (= $\mathbf{exp}(\boldsymbol{i}y)$) = cos y + $\boldsymbol{i}$ sin y so we come to one of the most important relations in all of mathematics:

$$e^{(x\ +\ \boldsymbol{i}y)} = e^{x}(\cos y + \boldsymbol{i}\sin y)$$

This is known as Euler's formula and it allows us to write any complex number not just as the sum of its real and imaginary parts but as the product of its modulus r and an exponential function of its argument θ:

$$\boldsymbol{z} = r\, e^{\boldsymbol{i}\theta}$$

Now that we know how to calculate e^z we can plot its map:



The z plane *The map of* $\mathbf{w} = e^z$

As you move the test object (here a letter F) vertically upwards, the image of the letter rotates round and round the origin, returning to its original position at every multiple of 2π. As you move the letter to the right, the image moves radially away from the origin according to an exponential formula. Essentially the X coordinate becomes the modulus and the Y coordinate becomes the argument of the image. This is why the whole of the real axis in the z plane is mapped onto the positive real axis of the w plane. Likewise, the imaginary axis of the z plane is mapped into a circle round the origin. The unit circle becomes rather distorted but note that angles are preserved because, like all functions with a polynomial expansion, it is analytic and hence conformal.

Now we have decided what we mean by $e^{i\theta}$, we need to define what we mean by things like $2^{i\theta}$ and even $z^{i\theta}$ where z is, of course, complex.

Now according to the ordinary rules which apply to exponents,

$$a^x = b^{x \log_b(a)}$$

As usual we want the same rules to apply to complex numbers so we will define

$$a^z = e^{z \log(a)}$$

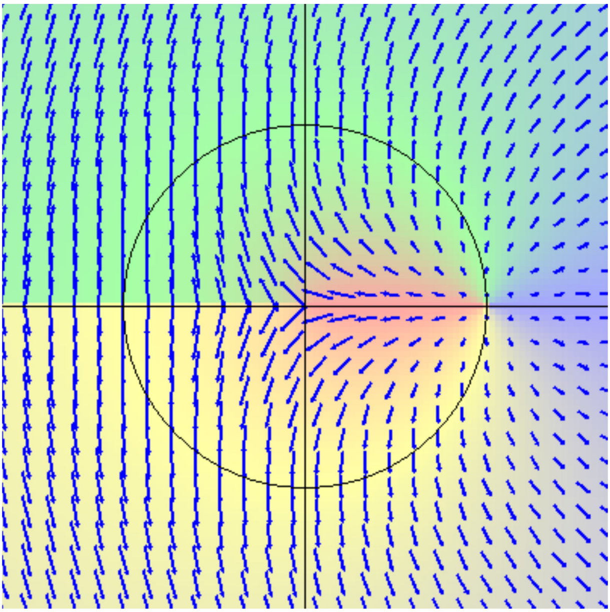
This is perfectly straightforward but if a is itself a complex number $\boldsymbol{\omega}$ then we have a slight problem. Obviously we require that:

$$\boldsymbol{\omega}^z = e^{z \log(\boldsymbol{\omega})}$$

but that means that we have to define $\mathbf{log(\omega)}$

Now if $\boldsymbol{\omega} = r\,e^{i\theta}$ then in order to be consistent, $\mathbf{log}(\boldsymbol{\omega}) = \log(r) + \boldsymbol{i}\theta$. i.e. the (logarithm of the) modulus of $\boldsymbol{\omega}$ becomes the real coordinate and the argument becomes the imaginary coordinate. This is, of course, the inverse of the exponential function.

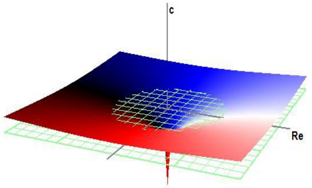
What does it look like? Well, the first thing to say is that the complex logarithm is a multi-valued function. We saw that the exponential function winds the imaginary axis round and round the origin an infinite number of times. It follows that there are an infinite number of values of y in the $\boldsymbol{\omega}$ plane which map onto the same argument of $\mathbf{log(\omega)}$. If we restrict ourselves to the values of y between $-\pi$ and $+\pi$, This is what we get.



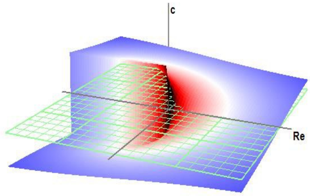
The vector plot of $\mathbf{w} = log(\boldsymbol{\omega})$

The first thing to notice is that it has a root at (1, 0). This is because the logarithm of 1 is zero. More importantly, it has a branch point at the origin and, unlike the branch points we have met before, this one is the meeting point of an infinite stack of discs!

The real and imaginary plots of **log**(**ω**) look like this:



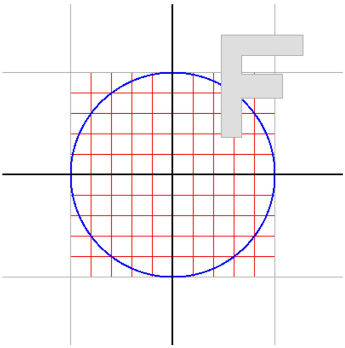
The real part of **log**(**ω**)



The imaginary part of **log**(**ω**)

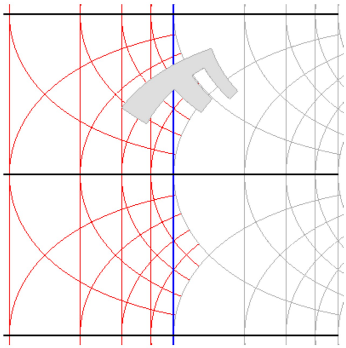
Note that, as you would expect, the function has a singularity at the origin but that it is perfectly well behaved everywhere else. The imaginary plot is a continuous helical spiral.

Its map looks like this:



I

The z plane

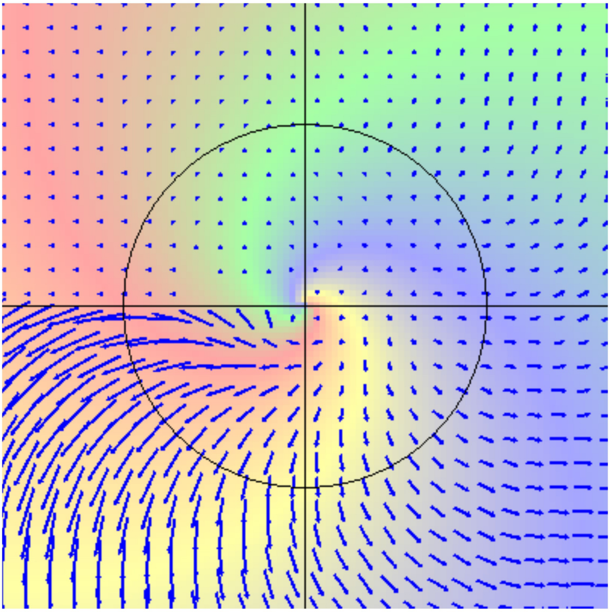


The map of ***w*** = **log**(**ω**)

The exponential function mapped the ***z*** plane multiple times round and round the origin; the logarithm function undoes this turning the unit circle back into the imaginary axis and mapping everything inside the unit circle onto the negative half of the ***w*** plane.

Note too that while log(–1) is undefined, there is no problem with the complex version **log**(–1). It is (0, (2*n* + 1)***i***) where *n* is any integer.

Lets finish this chapter with an example of a function in which a complex variable is raised to a complex power. For example the function $\boldsymbol{w} = \boldsymbol{z}^{(1 + \boldsymbol{i})}$.



The vector plot of $\boldsymbol{w} = \boldsymbol{z}^{(1 + i)}$

Don't ask me to explain it – just marvel at it!

The Zeta Function

No book on complex functions would be complete without a mention of one of the most extraordinary functions in all of mathematics – the Riemann zeta function. It is a function of a complex variable and is defined in the following way:

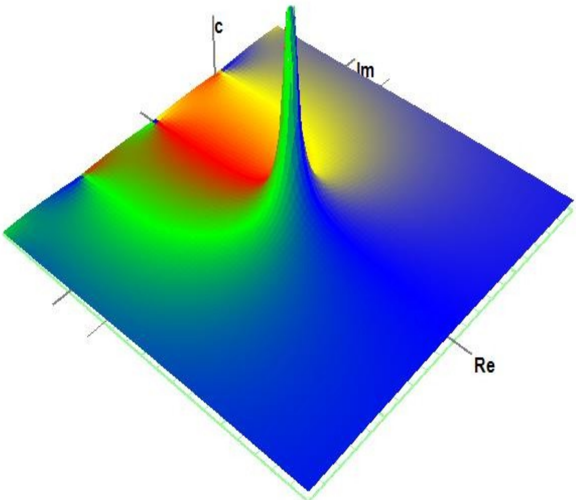
$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} \ldots$$

For example, if **z** = 2 then

$$\zeta(\mathbf{2}) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots$$

which turns out to be equal to $\pi^2/6$ or 1.6449...

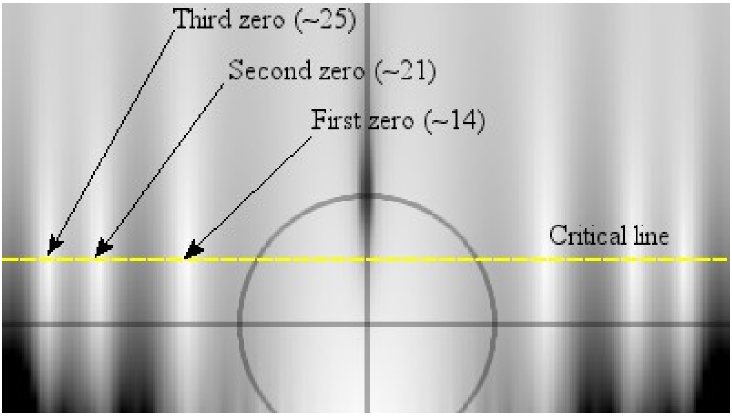
Calculating the values when **z** is some specific complex number turns out to be pretty complicated and special techniques are required for cases where the real value of **z** is less than 1 but it can be done and the result is illustrated below.



The modulus of w = **zeta**(*z*)

The illustration above shows the modulus (in 3D and the argument (in colour) in the region -1 < **Re**(*z*) < 3 and –2 < **Im**(*z*) < 2. It is obvious that the function has a singularity at the point (1, 0). (This is because the sum of the infinite series 1/1 + 1/2 + 1/3 + 1/4 … is infinite.)

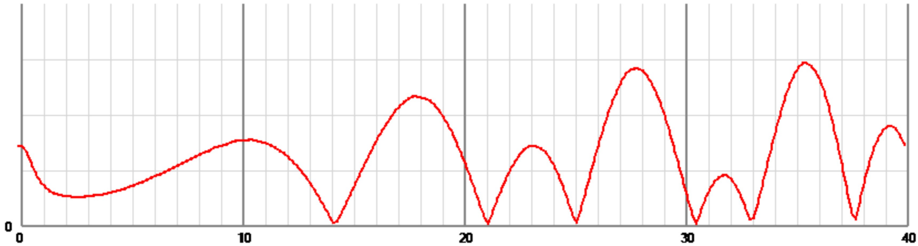
If we plot the function using a smaller scale on the imaginary axis (now plotted horizontally) we see that the modulus plot actually consists of a series of ridges and furrows parallel to the real axis.



The modulus of w = **zeta**(*z*) *with the imaginary axis multiplied by 10*

The singularity is the black area at (1, 0). Almost everywhere else, the modulus of the function is small but finite except at a series of points along the so-called 'critical' line where the real part of z is ½ (indicated by the yellow dotted line). At these points (known as the 'non-trivial zeros') the modulus is zero. The first of these zeros occurs at when the imaginary part is approximately 14. (The function is symmetrical about the real axis so there is another zero at (0.5 -14)). The second at approximately 21 and the third at 25.

You can see this a lot more clearly if we just plot a graph of the modulus along this critical line.



A graph of the modulus of the zeta function along the line (0.5, y)

The points where the modulus 'touches down' appear to be distributed quite randomly and there is no simple formula which will tell you where the n^{th} zero will fall. Laborious computer calculations are needed to locate the precise positions of the zeros. Indeed, the first mystery is why there are any zeros at all.

We do, however know that any zeros that do exist must lie in the 'critical strip' between $x = 0$ and $x = 1$ and of the millions of zeros which have been calculated, every single one lies exactly on the critical line where $x = ½$. The famous 'Riemann hypothesis' is simply the assertion that *all the zeros of the zeta function lie on the critical line*.

Riemann's hypothesis was included in Hilbert's famous list of unsolved problems in 1900. It remains unproved to this day. Mathematicians can be divided into three camps. There are those who firmly believe that the hypothesis is provable and that one day they will have the proof. Certainly there is a lot of evidence that the hypothesis is true. Using modern computers the positions of the first 2 million zeros have been calculated. In addition, many zeros way beyond this have been calculated, so far in fact that on a scale of 1mm to the unit, the furthest zero would be over a light year away – and all have so far been found to lie on the line. But specific examples do not constitute a proof. They do, however, give many mathematicians encouragement that there is some deep reason *why* all the zeros lie on the line and that it should not be beyond the wit of man to discover it.

On the other hand, the wit of man has been unequal to the task for 150 years. This has caused another group of mathematicians to speculate that, perhaps, the Riemann hypothesis is one of those famous

Gődel statement that are true but unprovable. If so the situation is similar to the situation facing geometers in the early 19^{th} century. Euclid's famous 'parallel postulate' was proving to be unprovable on the basis of the other axioms of the system – but 'obviously' true. Then came Lobochevsky who showed that it was perfectly possible to construct an alternative geometry in which the parallel postulate was false. What everyone had assumed up to that point was that geometry had to take place on a 'flat' surface. Could it be that there is some axiom of logic at the basis of number theory which we have so far unwittingly assumed which makes the Riemann hypothesis true but unprovable? And could it be that a future Lobochevsky will identify this axiom and thereby generate a new mathematics which contains all the familiar theorems of arithmetic but in which the Riemann hypothesis is false? Somehow I find this idea very unpalatable. Surely either the hypothesis is true – in which case there must be a *reason* why it is true; or it is false, in which case it is worth continuing the search for a counter-example.

There is a third option. The Riemann hypothesis is true and a proof of the hypothesis exists but the proof is so complicated that it will forever remain beyond the wit of man to devise it. If this is, in fact, the case, we shall have to rely on artificial intelligence to generate the proof for us. Even then, we will not necessarily be able to understand or check the proof which the computer has generated for us.

What is certain is that anyone who proves or disproves the Riemann hypothesis is destined for lasting fame.

References

The inspiration for this book came from Tristan Needham's excellent volume 'Complex Visual Analysis' OUP 1997

Almost all the images in this book were produced using the author's own software, much of which can be found on his web site:

www.jolinton.co.uk